Genus and other graph invariants

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Outline

Surfaces

A **surface** is a connected compact 2-manifold without boundary.

- A *sphere*, a *torus*, a *projective plane*, or a *Klein bottle* is a surface. The plane can be considered as a punctured sphere.
- We have two types of closed surfaces, either **orientable** or **non-orientable**.

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The classification theorem:

- The orientable surface S_g ($g \geq 0$) can be obtained from a sphere with $2g$ pairwise disjoint holes attached with g tubes (handles) such that each tube welds two holes.
- The number g is called the **genus** of the orientable surface.
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The classification theorem:

- The non-orientable surface N_k can be obtained from a sphere with k pairwise disjoint holes attached with k Möbius strips such that each Möbius strip welds one hole.
- The number k is called the **non-orientable genus** of N_k.
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Genus of graphs

- A graph is **embeddable** into a surface if we can draw the graph on the surface without edge-crossing.
- The **genus** of a graph G, denoted $\gamma(G)$, is the *minimum* g of S_g into which G is embeddable.
- A graph G is **planar** iff $\gamma(G) = 0$.

Genus of graphs

- Similarly, the **non-orientable genus** of a graph G, denoted $\bar{\gamma}(G)$, is the minimum k of N_k into which G is embeddable.
- A graph G is *projective-planar* iff $\bar{\gamma}(G) = 1$.

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Motivation

Only planar graphs ($\gamma = 0$) and projective planar graphs ($\bar{\gamma} = 1$) have been characterized by using minimal non-embeddable graphs (also called *excluded minor minimals* or *obstructions*).

The minimal non-embeddable graphs for torus ($\gamma = 1$) is unknown (more than 16,000).

Motivation

- *Contracting a non-loop edge* e is to delete e and then to identify the endpoints of e.
- A graph H is a *minor* of G if H can be obtained from a subgraph of G by contracting an edge repeatedly.
- We say that G *contains* H*-minor* if G contains a minor isomorphic to H.

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Known results

- A graph G is planar ($\gamma = 0$) $\iff$ G is K_5- and $K_{3,3}$-minor-free.
- G is $K_{3,3}$-minor-free toroidal ($\gamma = 1$) $\iff$ G is F_i-minor-free with $1 \leq i \leq 4$. (F_1 and F_2 are 0-sum and 1-sum of two K_5's, respectively.)

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- There are several methods to be considered to characterize genus of graphs.

  1. Use H-minor-free graphs for some small graphs H.
  2. Combine with other graph invariants: thickness, outerthickness or else?

- Later we introduce a graph-surface invariant.

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  1. Use H-minor-free graphs for some small graphs H.
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- Later we introduce a graph-surface invariant.

Motivation

- Edge-decomposition into forests (forest thickness) has been solved completely (Nash-Williams, 1964). It was called **arboricity**.

The **thickness** of a graph G, denoted $\Theta(G)$, is a minimum number of layers required for G to be decomposed into planar subgraphs.

Motivation

A graph G is **outerplanar** if and only if G is embeddable in the plane in such a way that all vertices of G are on the boundary of the outer-face.

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- In 2005, Daniel Gonçalves proved every planar graph can be decomposed to at most two outerplanar subgraphs.
  - Let $\mathcal{O}(t)$ be the class of all graphs with outerthickness at most t.
  - Every planar graph is in $\mathcal{O}(2)$.
  - For every graph G, $\Theta_o(G) \leq 2\Theta(G)$.

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Our preliminary results

- In other words, the class of K_5-minor-free and $K_{3,3}$-minor- free graphs is in $\mathcal{O}(2)$.
- We were interested in seeing how larger class of this kind, such as $K_{3,n}$-minor-free graphs, falls in $\mathcal{O}(2)$.

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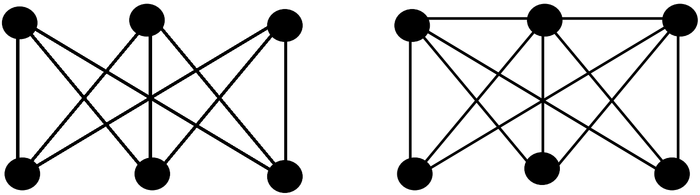
$K_{3,3}$ and $K_{3,3}^{++}$



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A new graph-surface invariant

- Let S be a surface $S_g (g > 0)$ or $N_k (k > 1)$. Suppose G is embeddable in S.
- We define a new graph-surface invariant, denoted **st(G,S)**.
- Let **e-curves** on S be disjoint simple closed noncontractable curves. The standard **meridian** on a torus is an e-curve.

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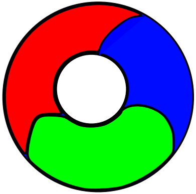
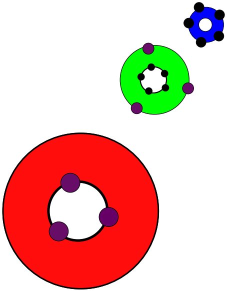
3 meridians on a torus



The torus consists of blue, green and red cylinders.

3 cylinders and vertices



Vertices on the 3 cylinders are appropriately identified.

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- Among all embeddings of G into S, let $st(G, S)$ be the minimum number of e-curves on S such that the curves pass through all vertices without crossing any edges of G.
- If G is toroidal, then $st(G, S_1)$ is called the **meridian number** of G.

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Advantages of the $st(G, S)$.

- Let $\mathcal{ST}(S, t)$ be the set of graphs G embeddable in S with $st(G, S) \leq t$. Then $\mathcal{ST}(S, t)$ is **topological-minor-closed**. Not minor-closed because contracting an edge whose endpoints are on different e-curves makes the e-curves intersect.
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The followings have meridian number 1.

- K_7 (David VanHeeswijk)
- $K_{4,4}$
- $K_{3,6}$
- $(K_5 - e) +_0 (K_5 - e)$
- $K_5 +_1 (K_5 - e)$

Stacey McAdams: $K_5 +_0 (K_5 - e)$ has meridian number at most 2.

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Conjecture (J.K.):
The toroidal graph obtained from p-fold covering space of (K_7, S_1) along the standard longitude of S_1 has meridian number p.

Remark: This is false for arbitrary graph. There exists a graph that its 3-fold covering space has meridian number 1.

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Problems

Here are unsolved problems.

- Is every projective-planar graph in $\mathcal{O}(2)$?
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- Find excluded minor minimals for each set of $\mathcal{ST}(S_1, t)$.
- Find excluded topological minor minimals for each set of $\mathcal{ST}(S_1, t)$.
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