Online Scheduling and Paintability

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Joint work with
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Goal: Consider an online version of choosability.

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Worst-case analysis is modeled by the following game:

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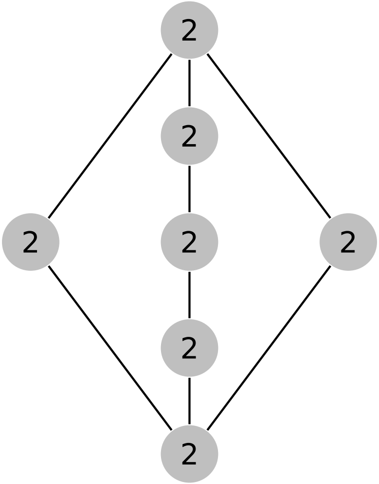
• An adaptive Lister, responding to Painter's earlier moves, may do better.

Example Game

Let's play the Lister/Painter game on $\Theta_{2,2,4}$.



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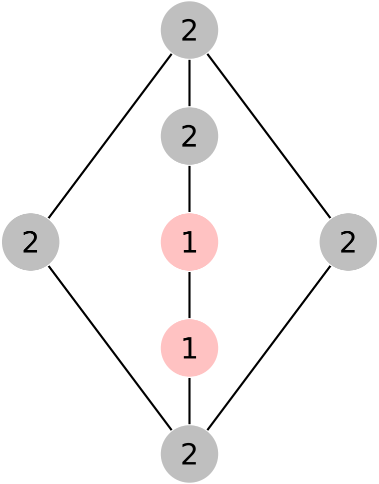

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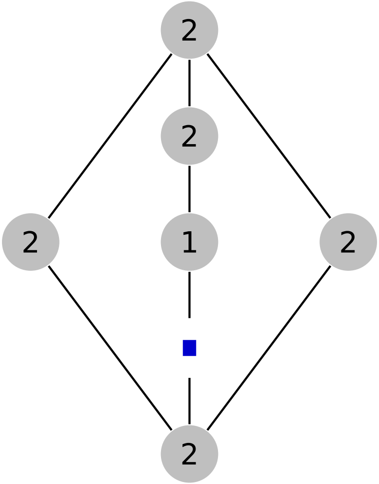

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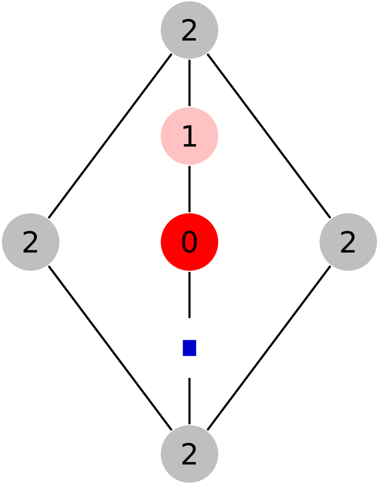

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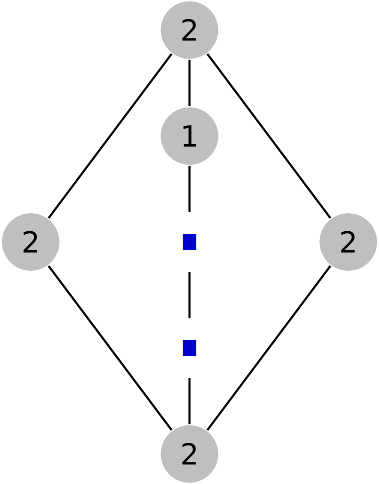

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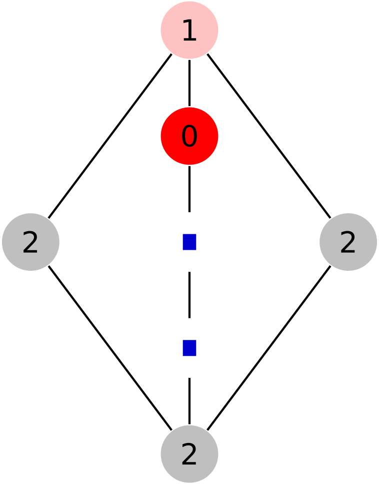

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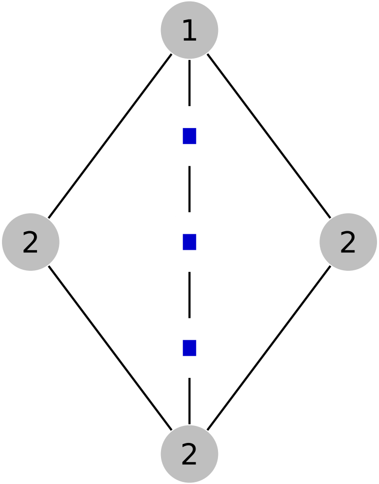

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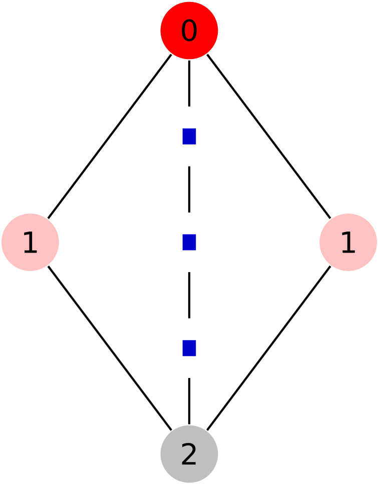

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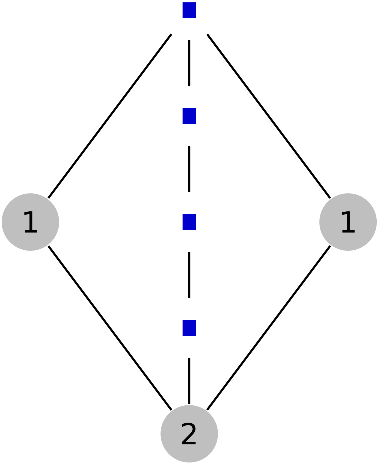

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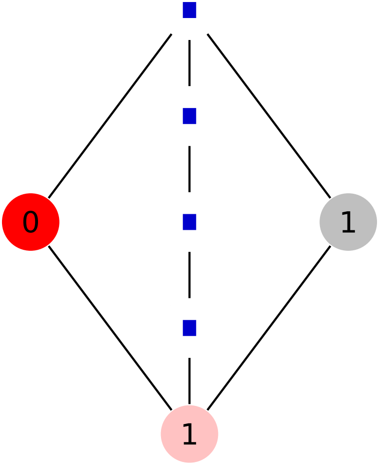

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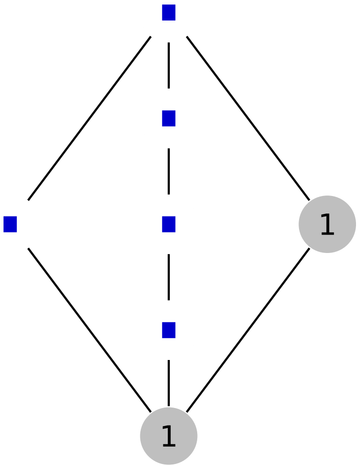

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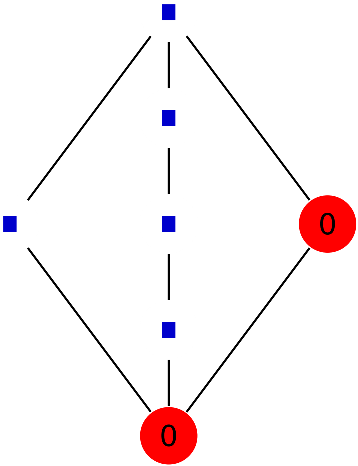

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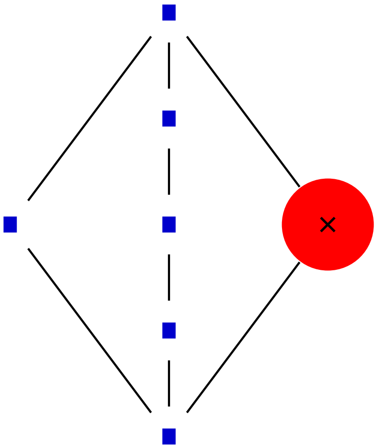

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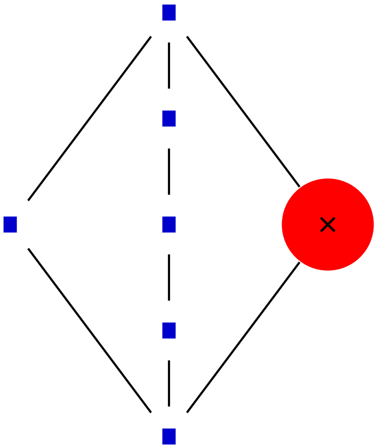

Conclude: Lister wins on $\Theta_{2,2,4}$ when each vertex has 2 tokens.

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When G is bipartite,

G is $\Delta(G)$-edge-colorable (König [1916])

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Tournament Scheduling (Schauz [2010])

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Thm. (CLMPTW) If G is k-paintable and $|V(G)| \le \frac{t}{t-1}k$, then $G \oplus \overline{K}_t$ is $(k+1)$-paintable.

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Thm. (CLMPTW) If G is k-paintable and $|V(G)| \le \frac{t}{t-1}k$, then $G \vee \overline{K}_t$ is $(k+1)$-paintable.

Pf. Idea: Painter uses a k-paintability strategy **S** on G, ignoring the added t-set T, until a special round where $M \cap T$ is colored instead. Each $v \in T$ has a token left, and G can be finished with the extra tokens in $V(G)$.

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Complete Bipartite Graphs

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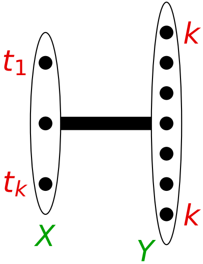
where $t_1, \ldots, t_k$ are the token counts in X.



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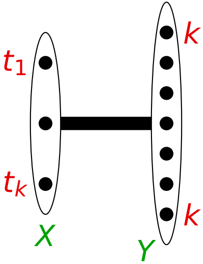

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k-paintability for $K_{k,r}$

Thm. (CLMPTW) Consider $K_{k,r}$ with $|X| = k$ and $|Y| = r$.
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Pf. $r = \prod t_i$ $\Rightarrow$ $K_{k,r}$ *is not f-choosable.*
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$|Y - M| < \prod t_i - q \le \prod_{i=1}^{k-1} t_i (t_k - 1)$; ind. hyp. applies! ■

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