Locating a Robber on a Tree.

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Original Cops and Robbers

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Game

1. *Cop chooses vertex*
2. *Robber chooses vertex*
3. *Cop and Robber take turns moving to adjacent vertices (visible to both players)*
4. *Cop wins if he ‘arrests’ the Robber*

Examples

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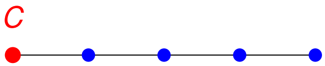

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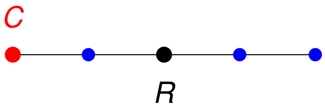

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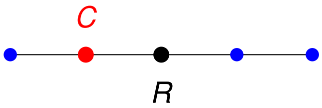

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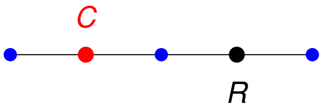

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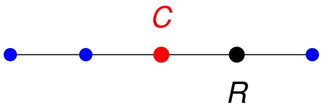

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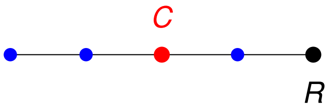

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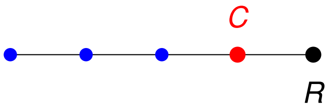

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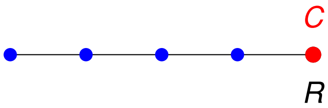

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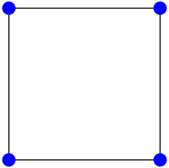

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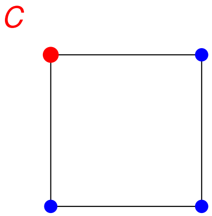

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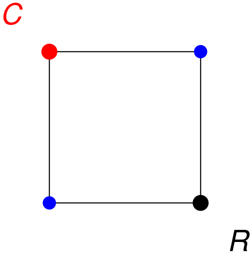

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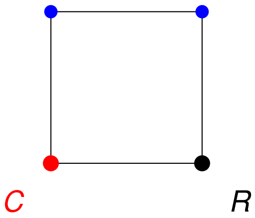

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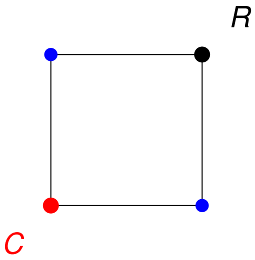

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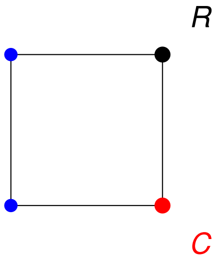

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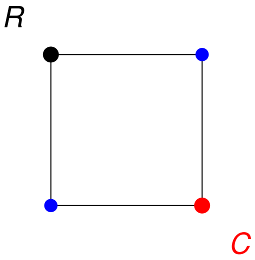

Motivating Questions

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*A graph is **cop-win** if there is a strategy under which the Cop arrests the Robber in a finite number of moves. Otherwise, a graph is **robber-win**.*

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Example

A tree T is cop-win.

1. Which graphs are cop-win?
2. For a graph G, what is the minimum number of Cops required to capture the Robber?
3. What is the minimum guaranteed capture time?

Early Results

Theorem (Nowakowski, Winkler 1983)

A finite graph G is cop-win if and only if G is dismantlable.

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Theorem (Seymour, Thomas 1993)

A graph G has treewidth $k - 1$ iff k is the minimum number of cops for which G is cop-win.

On Large Classes

- Directed graphs (Hahn, MacGillivray 2006)
- Edge Critical graphs (Clarke, Fitzpatrick, Nowakowski 2010)
- Forbidden (induced) subgraphs (Joret, Kamiński, Theis 2010)
- Interval graphs (Gavenčiak 2011)
- Geometric graphs (Beveridge, Dudek, Frieze, Müller 2012)
- Random graphs (Scott, Sudakov 2011; Bollobas, Kun, Leader 2013)

Mobility Variations

- Robber can Hide and Ride (Chalopin, Chepoi 2011)
- Lazy Robber (Richerby, Thilikos 2011)
- Fast Robber (Alon, Mehrabian 2011; Frieze, Krivelevich, Loh 2012)
- Drunk Robber (Kehagias, Prałat 2012)

Search Variations

- Helicopter Search (Fomin 1998)
- Tandem Search (Clarke, Nowakowski 2005)
- Reduced Visibility (Isler, Karnad 2008)
- Witness (Clarke 2009)
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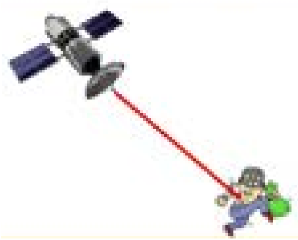

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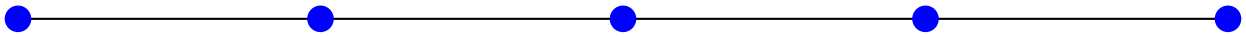
Definition

A graph G is ***locatable*** *if the Cop has a winning strategy.*

For locatable G, the ***location number*** *of G, $loc(G)$, is the minimum number of probes guaranteed to locate the Robber.*

Proposition (Seager 2012)

$loc(G) = 1$ if and only if G is a path.



P_5

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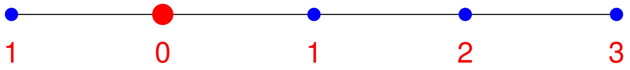

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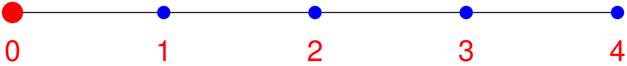

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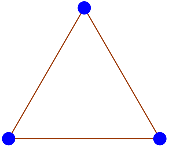

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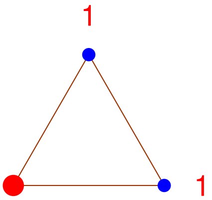

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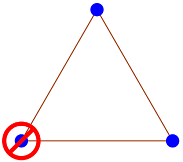

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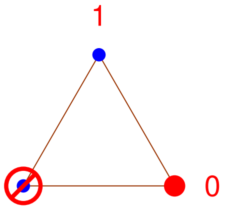

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Proposition

If G contains $K_{3,3}$ as an induced subgraph, then G is not locatable.

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C_n is locatable for $n > 5$. With, $loc(C_n) = 3$ for $6 \leq n \leq 11$ and $loc(C_n) = 2$ for $n > 11$.

Seager's Results For Trees

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$loc(K_{1,n-1}) = n - 2$ *for* $n \geq 3$.

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$loc(K_{1,n-1}) = n - 2$ for $n \geq 3$.

Theorem

If T is a tree with $n \geq 3$ vertices, then $loc(T) \leq n - 2$, with equality if and only if $T = K_{1,n-1}$.

r-Locating Strategy

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*An **r-locating strategy** locates the robber if he is ever at $r \in V(T)$.*
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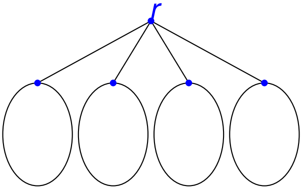

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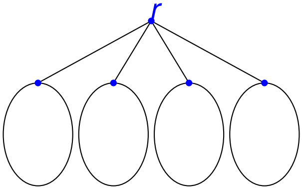

Remark

Under an r-locating strategy, the Robber will never be able to move from one component of $T - r$ to another.

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- Improve bound for $loc(T)$ using parameters other than n

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d=5 One arm is star of stars, each other arms is a star $\Rightarrow < \ell - 1$.

Main Result

Theorem (Brandt, Diemunsch, Erbes, LeGrand, M. 2013+)

If T is a tree with $diam(T) = d \geq 6$ and $\Delta(T) = \Delta$, then

$$loc(T) \leq (2\Delta^2 - 6\Delta + 4)\lceil \frac{d}{2} \rceil - 4\Delta^2 + 13\Delta - 9.$$

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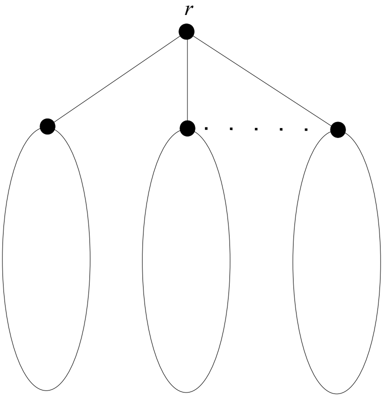
Answering Seager's Questions

- Algorithmic construction of a strategy.
- For $diam(T) \geq 6$, $f(\Delta, d) < n - 2$.
- Conjecture $loc(T) \leq \ell - 1$...?

Algorithm Phase 1:

- Pick a Root.



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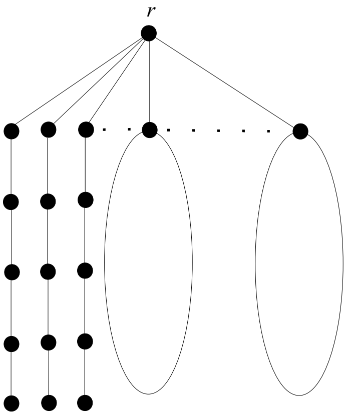

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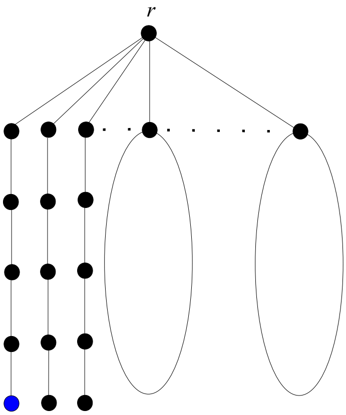

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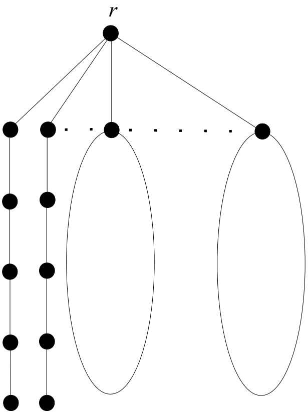

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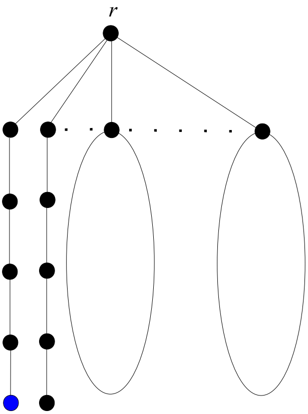

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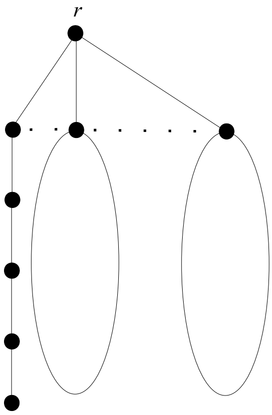

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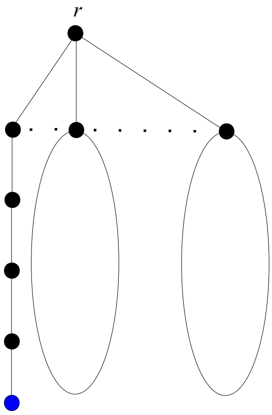

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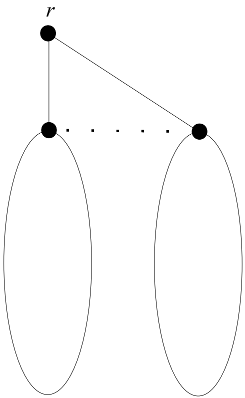

Algorithm Phase 1:

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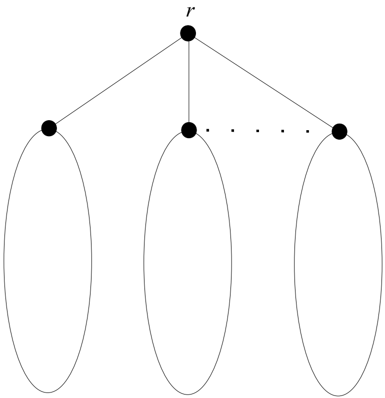
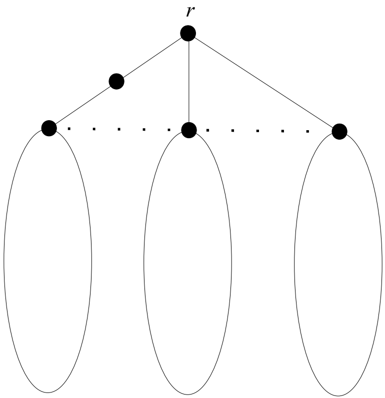


Algorithm Phase 1:

- Pick a Root.
- Remove paths.
- Remove Long Arms.



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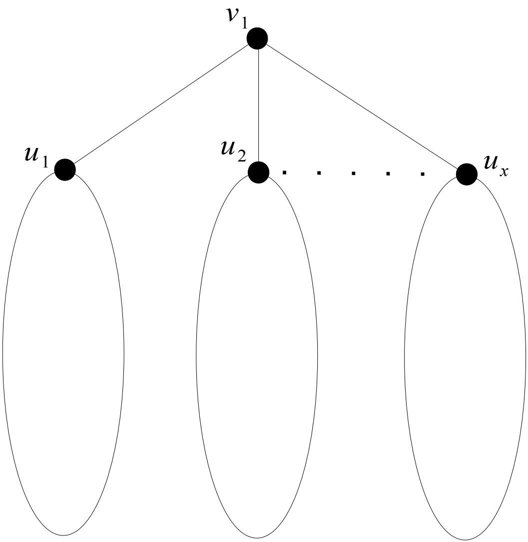

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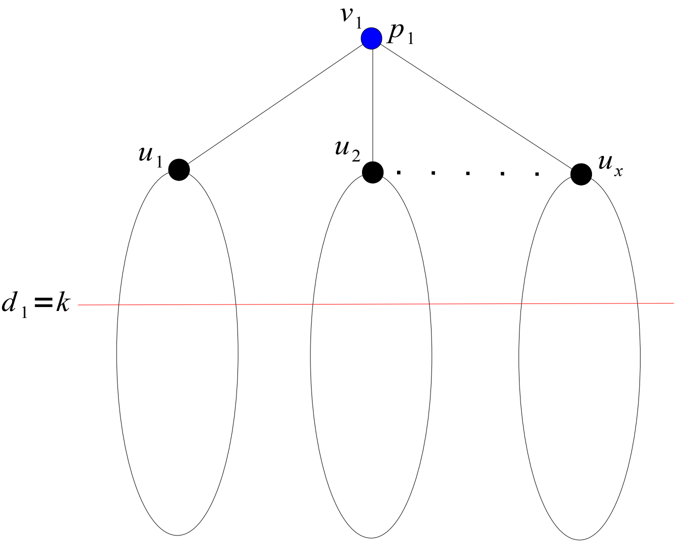

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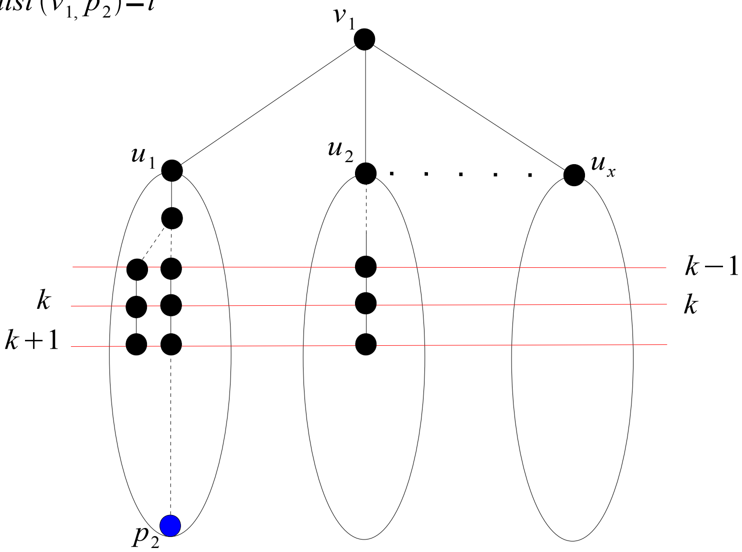

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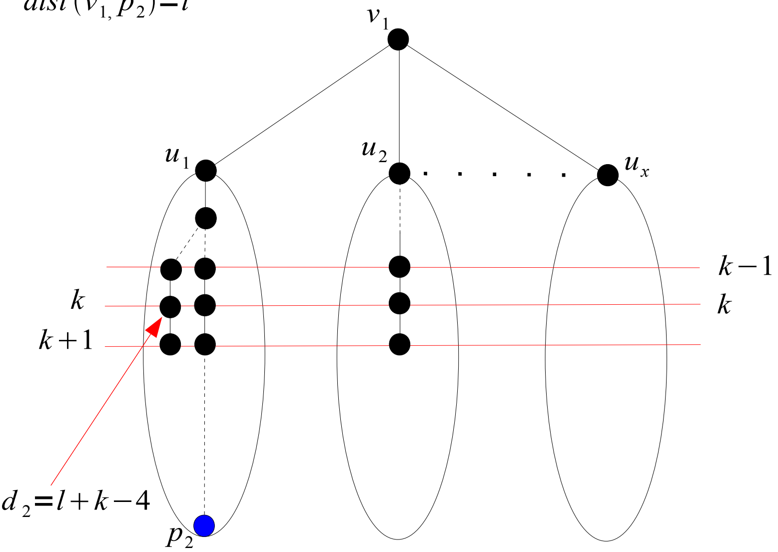

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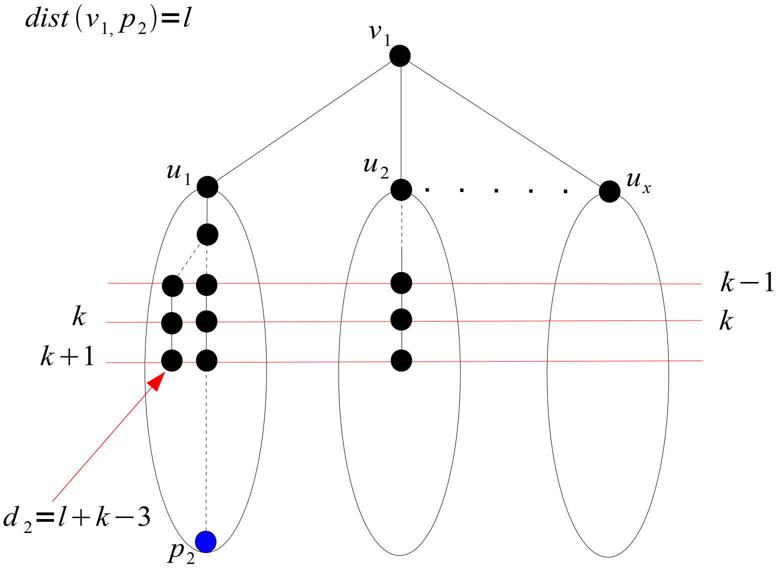

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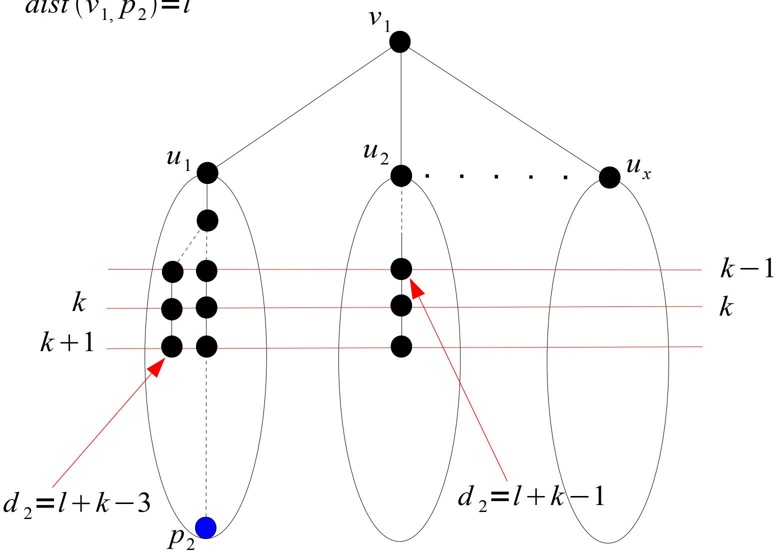
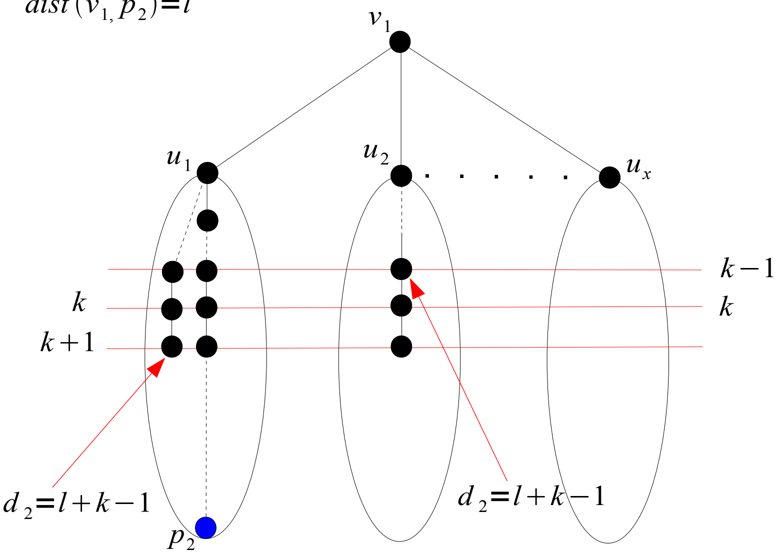


Algorithm Phase 3:

$dist(v_{1,} p_2) = l$



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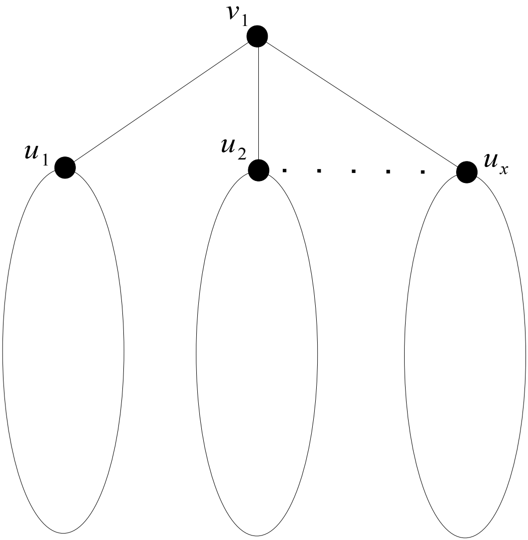

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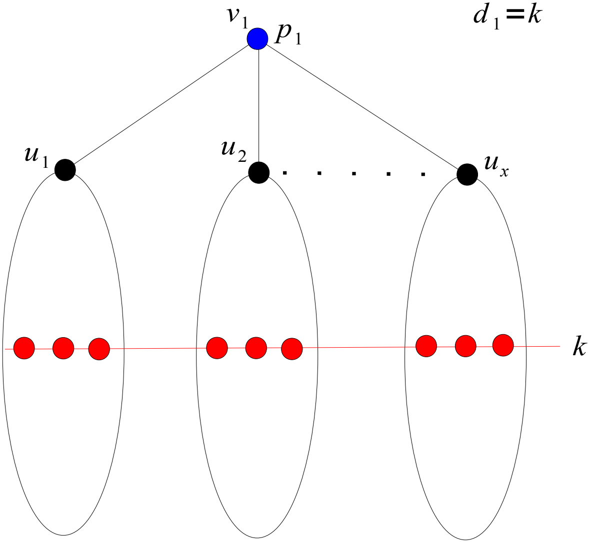

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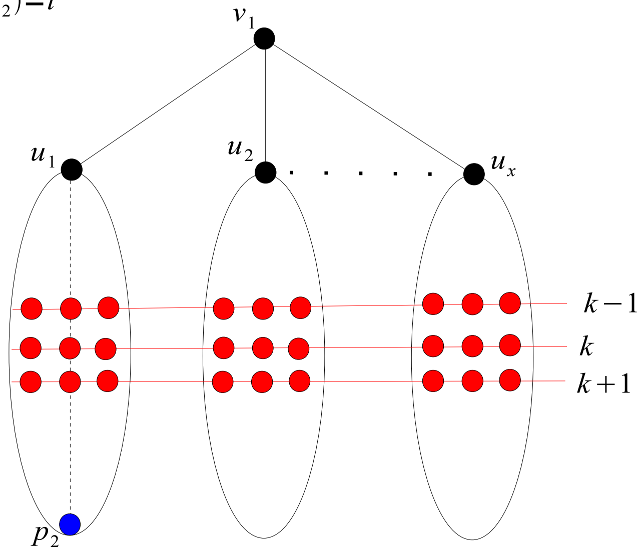

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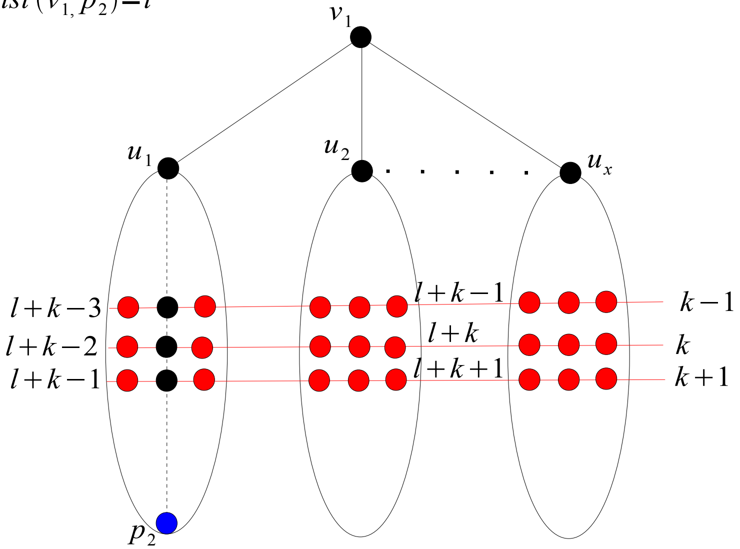

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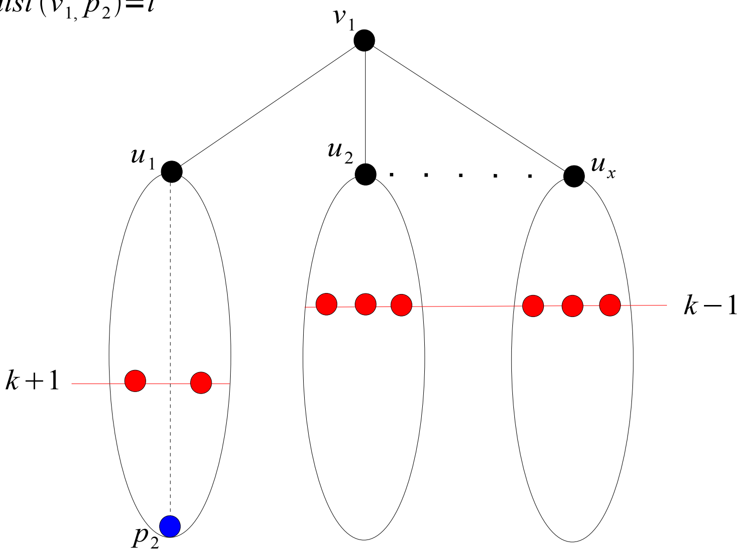

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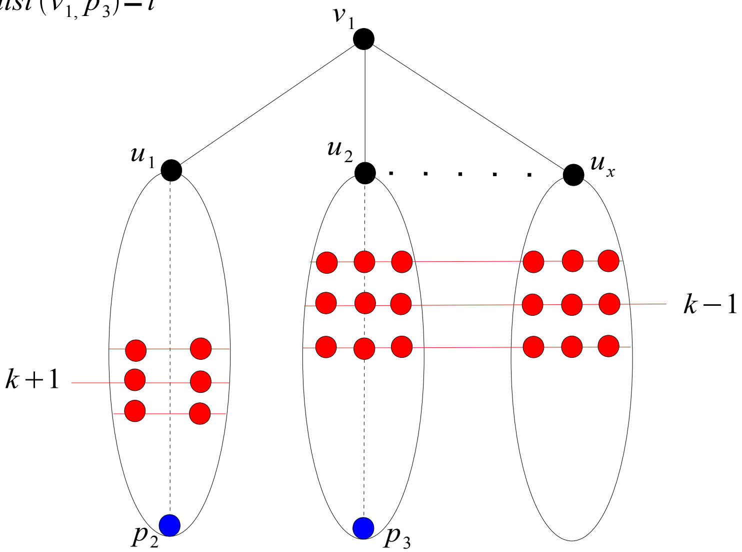
$dist(v_{1,} p_3) = l$



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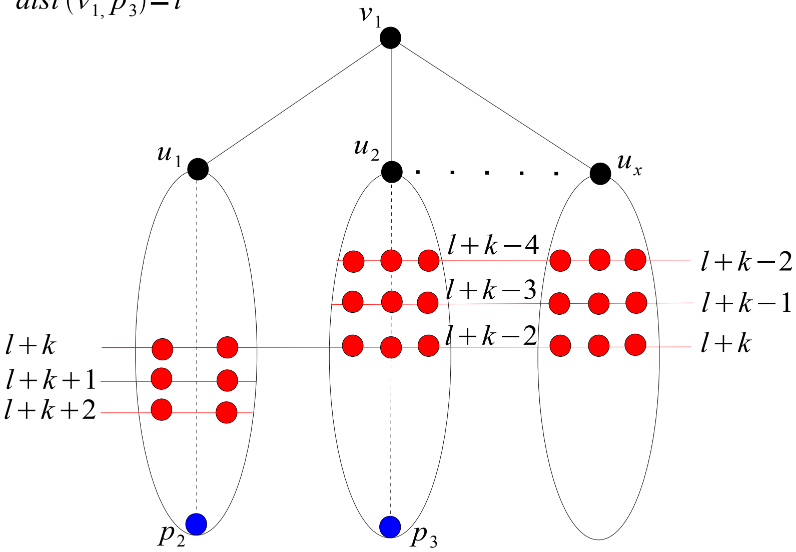

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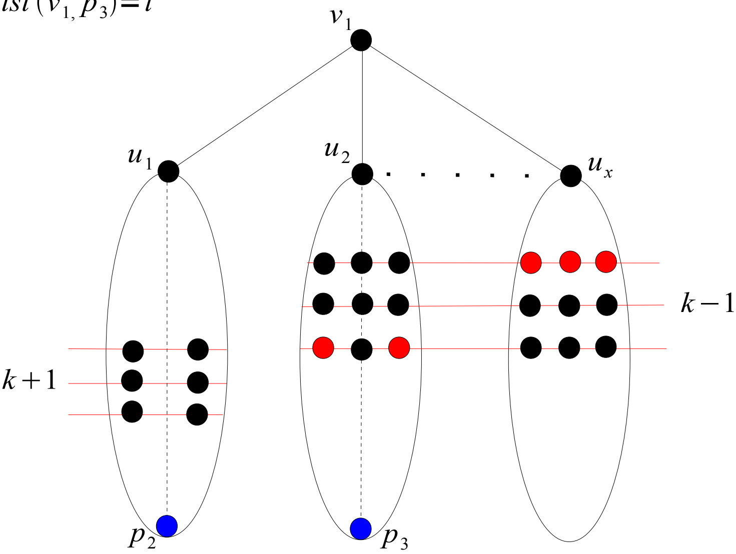

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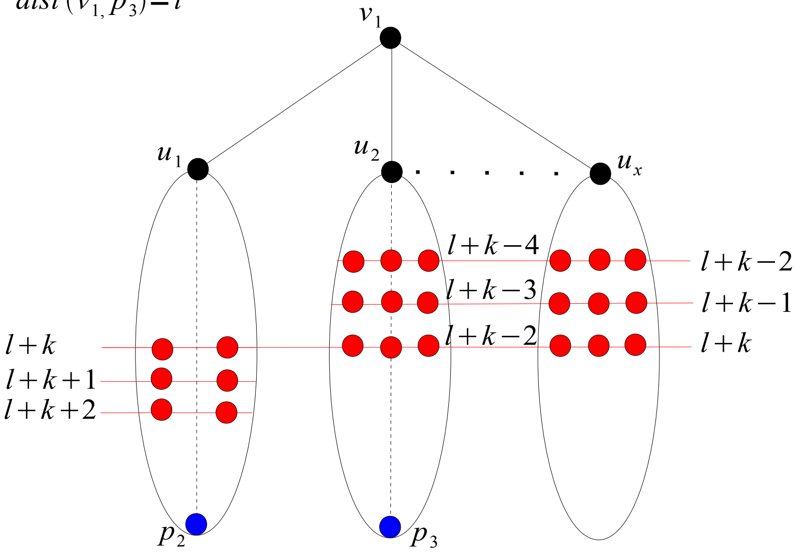

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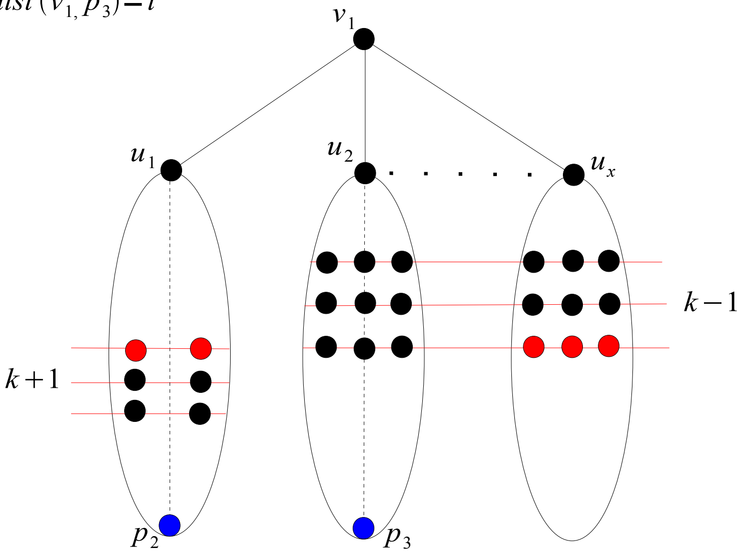

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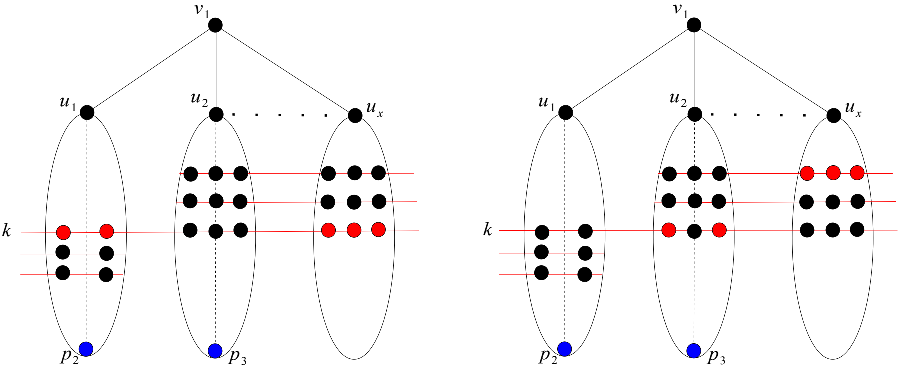

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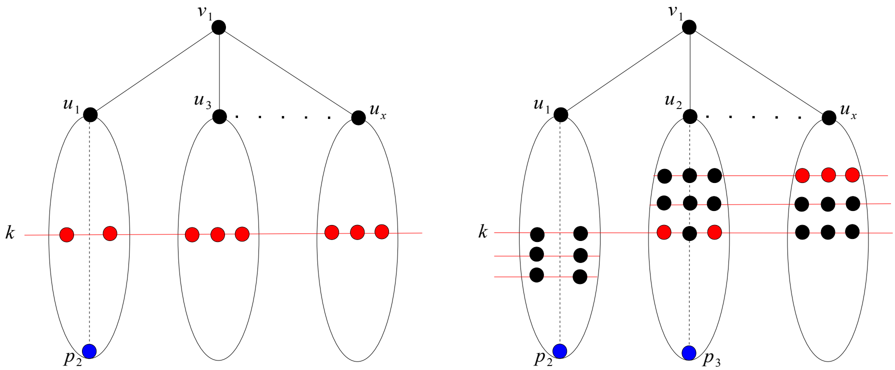

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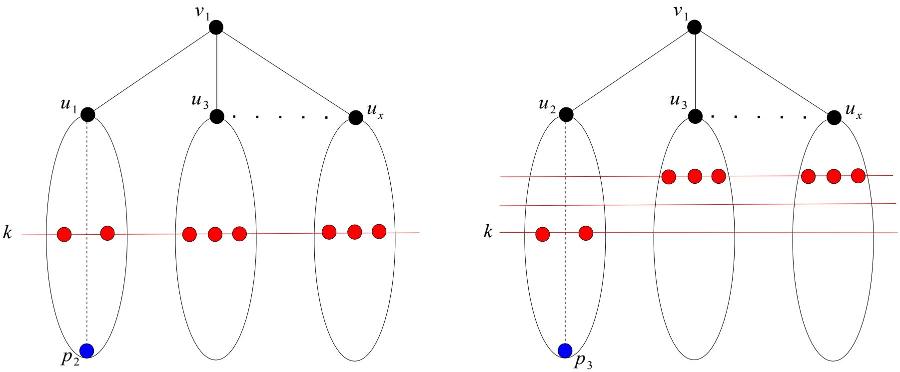

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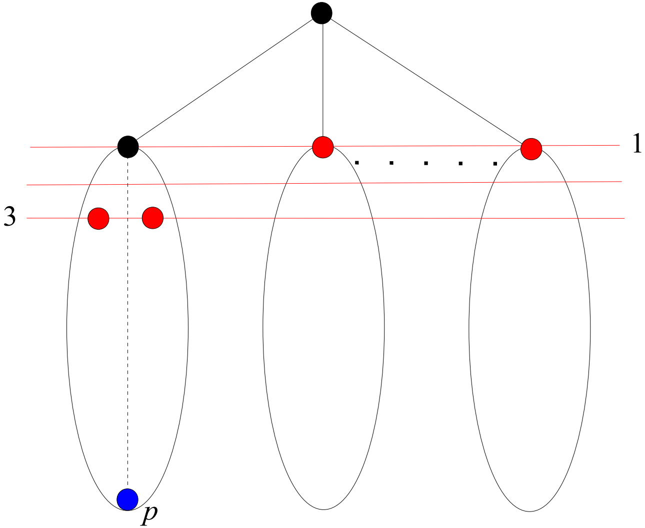

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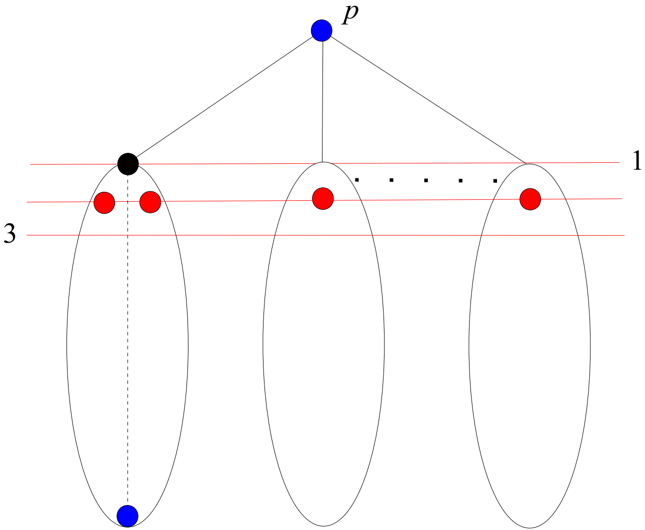

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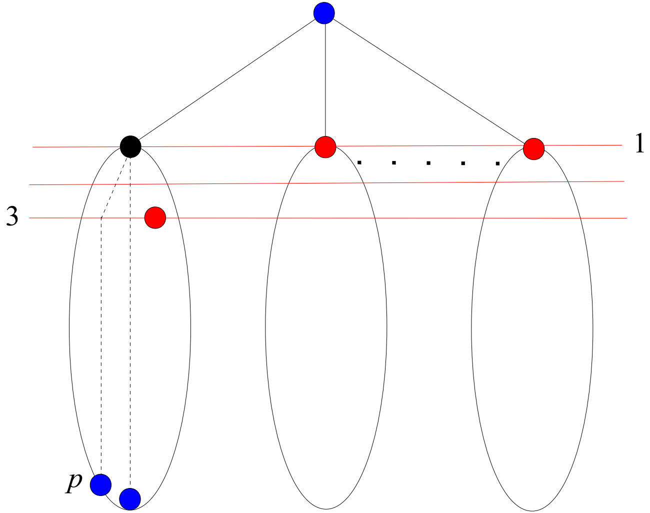

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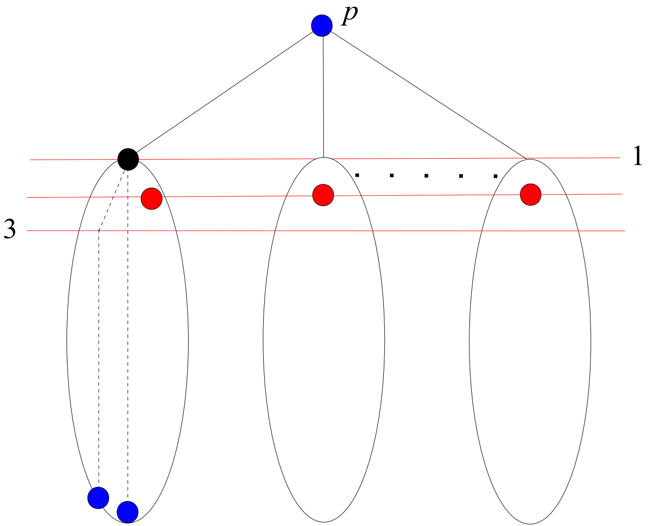

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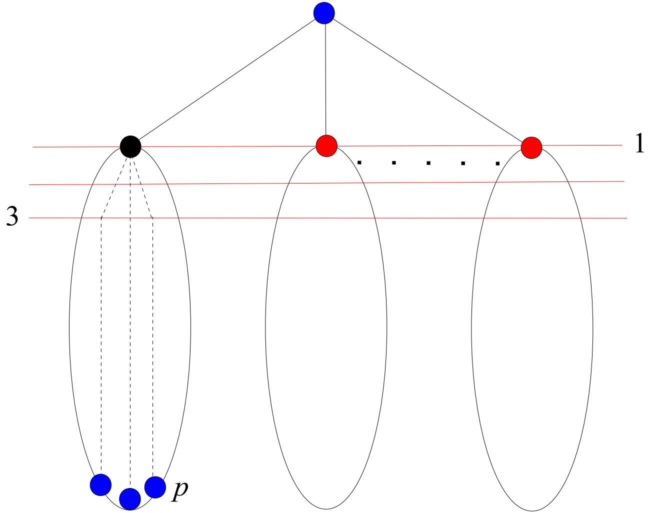

Algorithm Phase 4:

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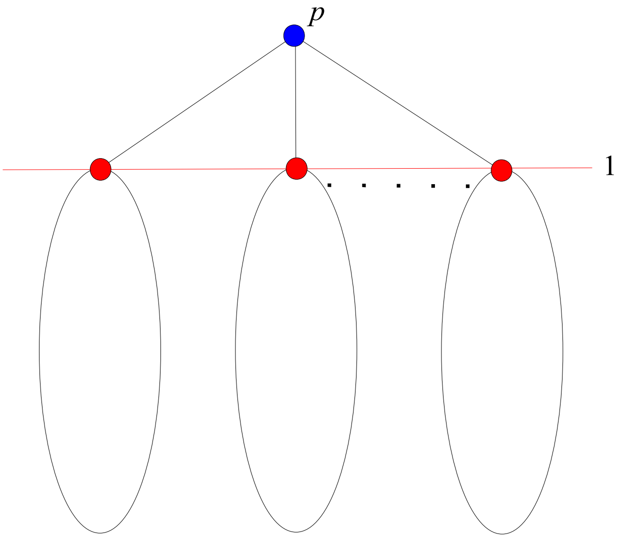

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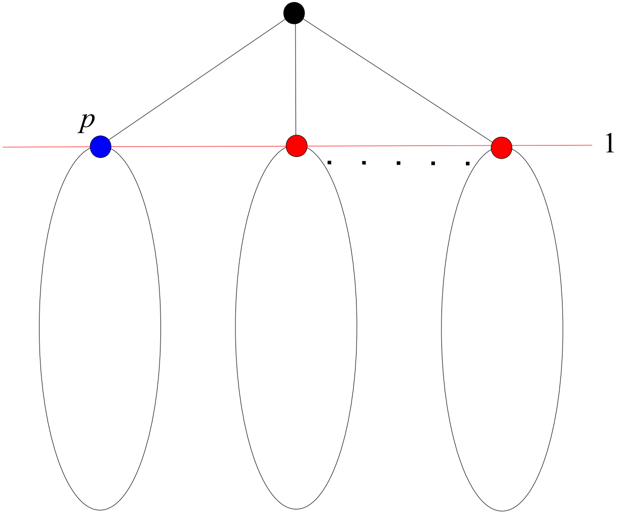

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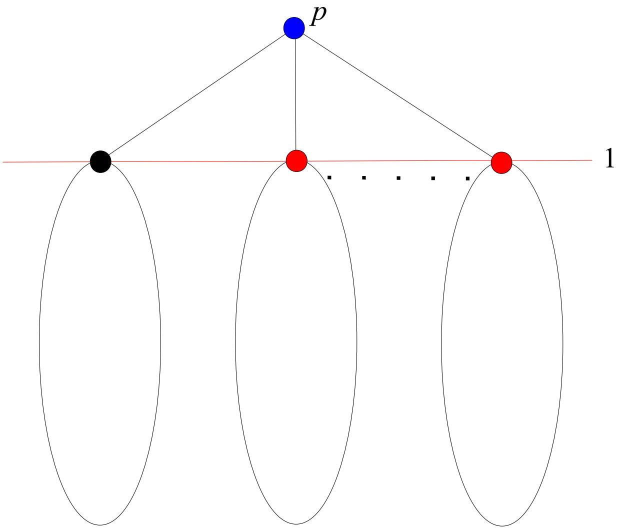

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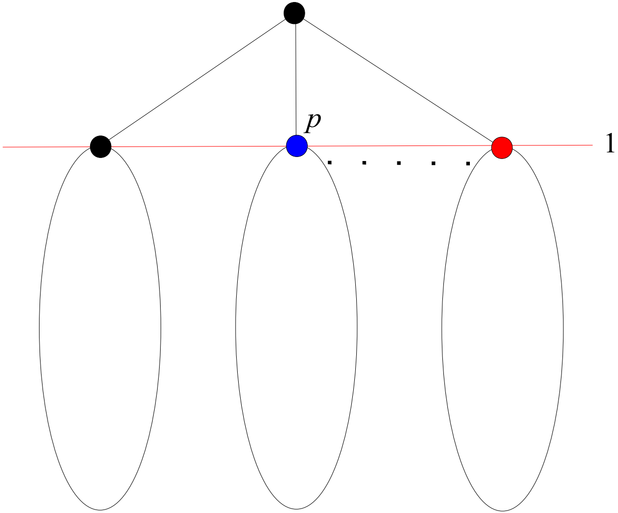

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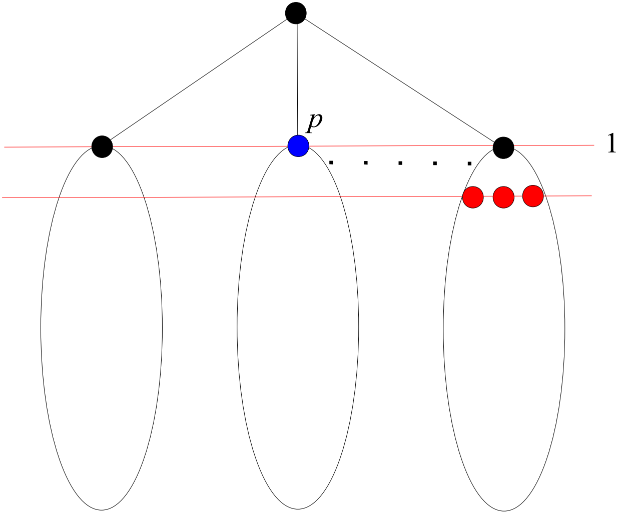

Algorithm Upper Bound

Lemma

If $T \subseteq T'$ *then* $loc(T) \leq loc(T')$

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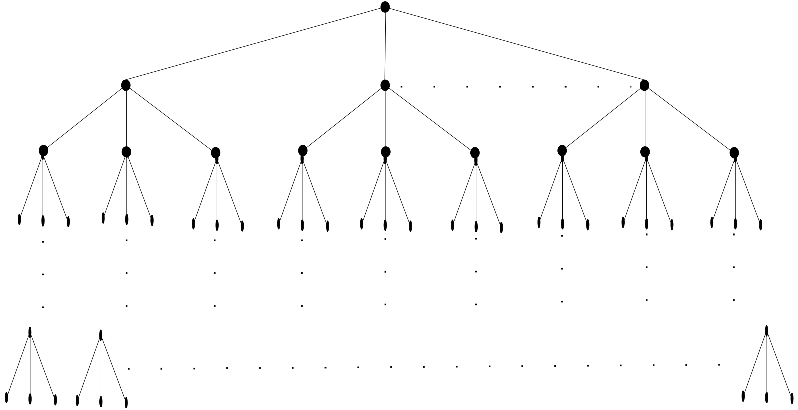
If $T \subseteq T'$ then $loc(T) \leq loc(T')$

- Let T' be the smallest Δ-inary tree such that $T \subseteq T'$.
- Determining $loc(T')$ gives an upper bound on $loc(T)$.

Algorithm Upper Bound (Diameter at least 6).



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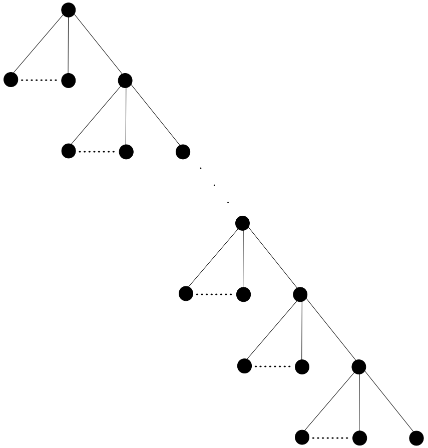

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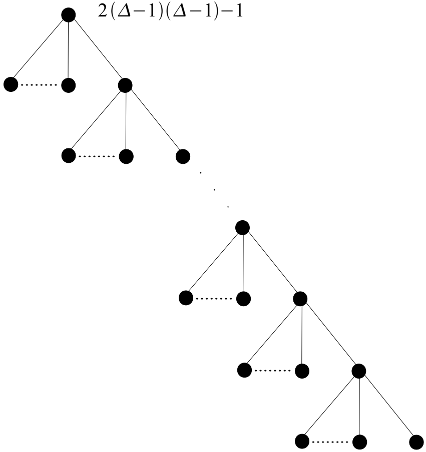

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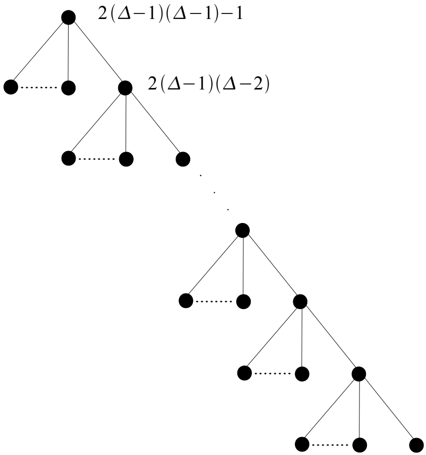

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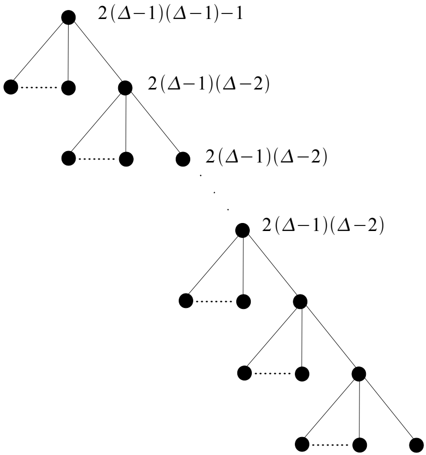

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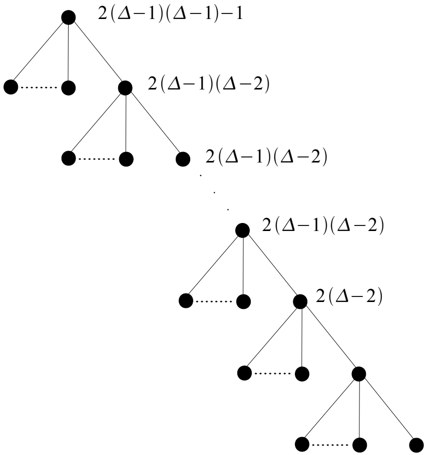

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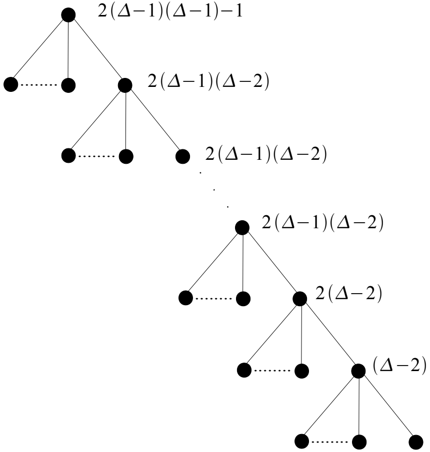

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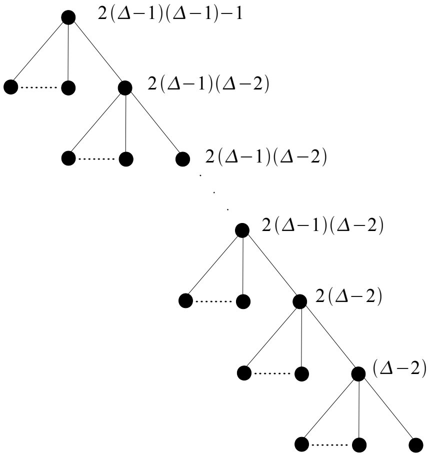
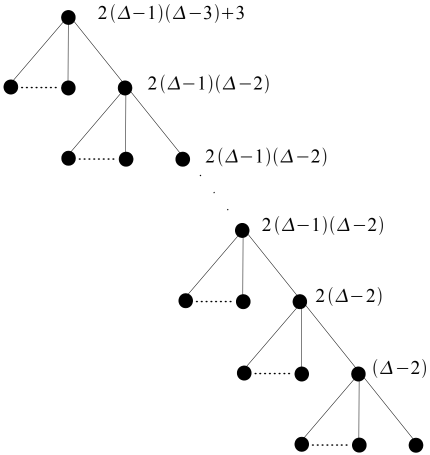


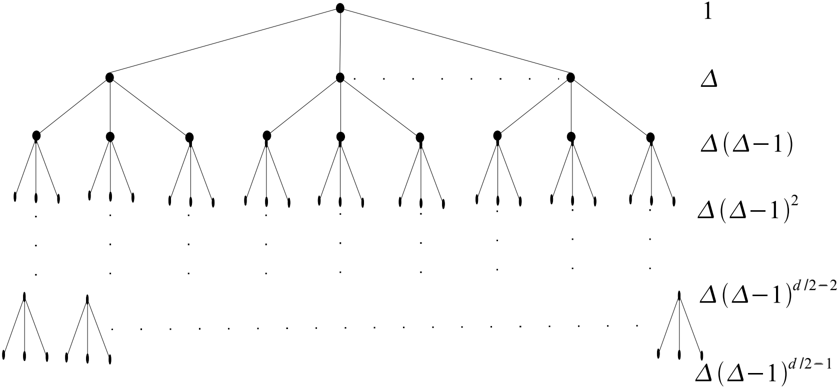
$$loc(T) \leq (2\Delta^2 - 6\Delta + 4)\lceil \frac{d}{2} \rceil - 4\Delta^2 + 17\Delta - 17$$

Algorithm Upper Bound (Diameter at least 6).



$$loc(T) \leq (2\Delta^2 - 6\Delta + 4)\lceil \frac{d}{2} \rceil - 4\Delta^2 + 13\Delta - 9$$

Compare with $n-2$ For k-inary Tree.



$$\sum_{i=0}^{d/2-1} \Delta(\Delta-1)^i + 1$$

Compare with $n-2$ For k-inary Tree.

$$f(\Delta, d) = (2\Delta^2 - 6\Delta + 4)\lceil \frac{d}{2} \rceil - 4\Delta^2 + 13\Delta - 9$$

- $f(\Delta, d) \leq d\Delta^2$

Compare with $n-2$ For k-inary Tree.

$$f(\Delta, d) = (2\Delta^2 - 6\Delta + 4)\lceil \frac{d}{2} \rceil - 4\Delta^2 + 13\Delta - 9$$

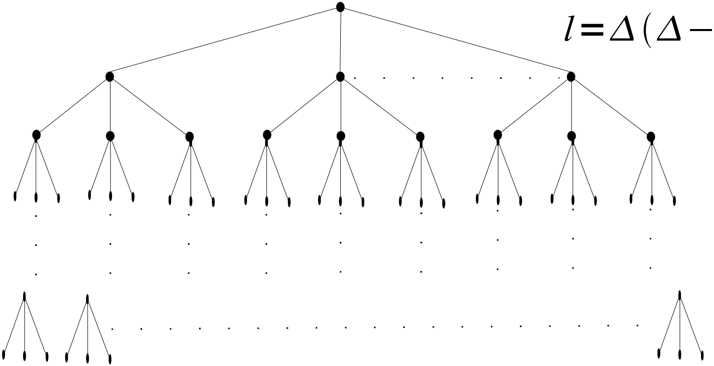
- $f(\Delta, d) \leq d\Delta^2$
- $d\Delta^2 < \sum_{i=0}^{\frac{d}{2}-1} (\Delta)(\Delta-1)^i - 1$

Compare with $n-2$ For k-inary Tree.

$$f(\Delta, d) = (2\Delta^2 - 6\Delta + 4)\lceil \frac{d}{2} \rceil - 4\Delta^2 + 13\Delta - 9$$

- $f(\Delta, d) \leq d\Delta^2$
- $d\Delta^2 < \sum_{i=0}^{\frac{d}{2}-1} (\Delta)(\Delta-1)^i - 1$
- So $f(\Delta, d) \ll n-2$ for Δ-inary tree.

Compare with $\ell - 1$...?



$$l = \Delta(\Delta - 1)^{d/2-1}$$

$$loc(T) \le (2\Delta^2 - 6\Delta + 4)\lceil \frac{d}{2} \rceil - 4\Delta^2 + 13\Delta - 9$$

Compare with $\ell - 1$...?

Phase 1: Each ping we delete one leaf.

Compare with $\ell - 1$...?

Phase 1: Each ping we delete one leaf.

Phase 2: Every two pings we delete at least two leaves.

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Phase 3: Each leaf takes at most two pings (except first/last).

Compare with $\ell - 1$...?

Phase 1: Each ping we delete one leaf.

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Phase 3: Each leaf takes at most two pings (except first/last).

Phase 4: Half of the leaves take at most two pings.

Compare with $\ell - 1$...?

Phase 1: Each ping we delete one leaf.

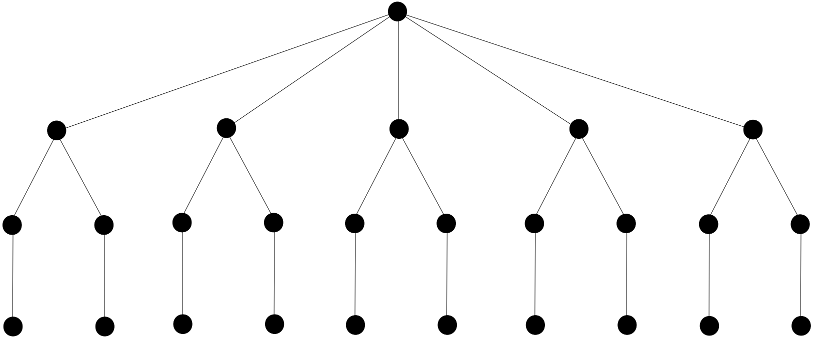
Phase 2: Every two pings we delete at least two leaves.

Phase 3: Each leaf takes at most two pings (except first/last).

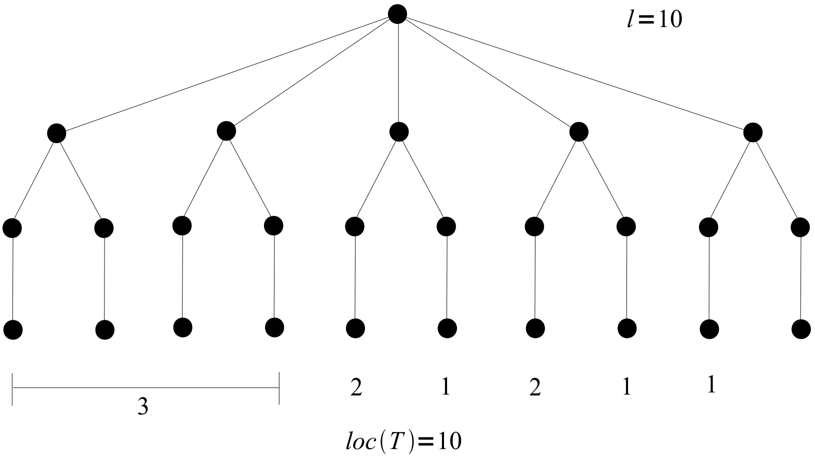
Phase 4: Half of the leaves take at most two pings.

$$\Rightarrow \mathit{loc}(T) < 2\ell.$$

Counter Example



Counter Example



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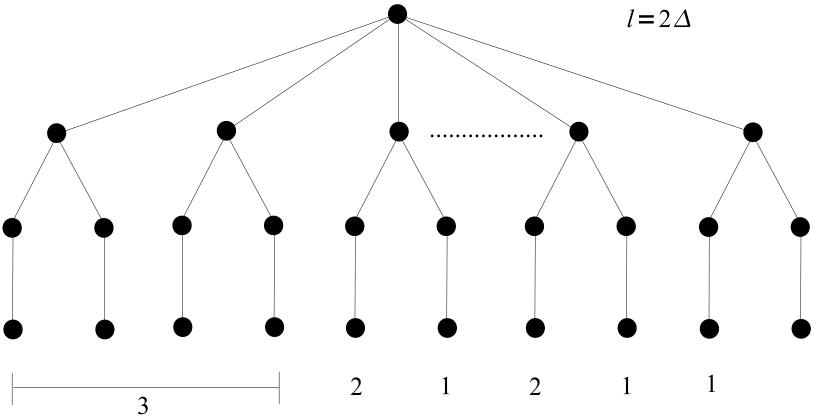
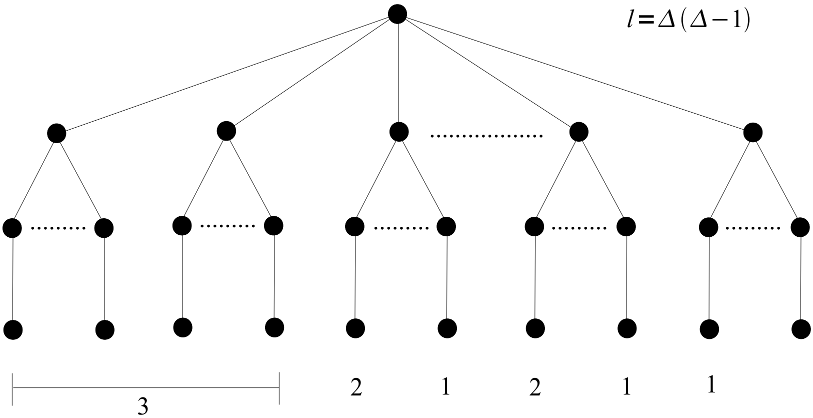


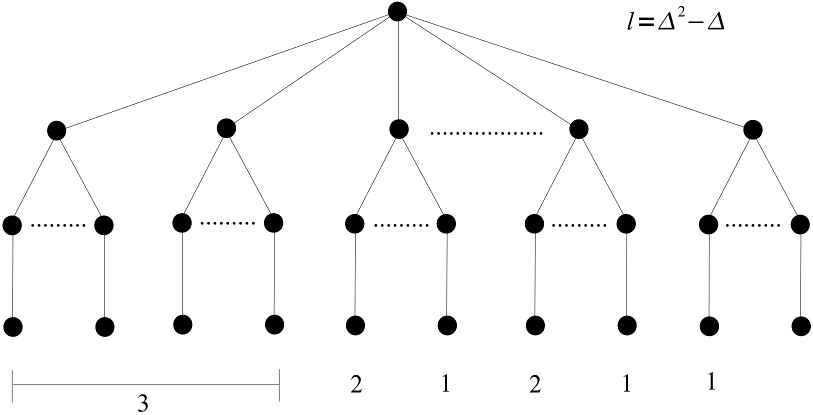
$loc(T)=3\Delta-5$

Counter Example



$$loc(T) = 3 + (2(\Delta - 2) - 1)(\Delta - 3) + 1$$

Counter Example



$$loc(T) = 2\Delta^2 - 11\Delta + 19$$

Future Work

Question

Lower Bound as a Function of ℓ?

Future Work

Question

Lower Bound as a Function of ℓ?

Question

Can we generalize this to larger classes?

Thank You

Thank You

Seager. **Locating a robber on a graph.** *Discrete Math.* 312 (2012), no. 22, 3265-3269.