# Investigating Chance

Unit 5

## Mathematical Concepts

- Probability is a measure of the likelihood of a chance event.
- Theoretical and empirical probabilities are estimates of the true probability of a chance event.
- Theoretical probability relies on analysis of the structure of the chance process generating the event. Theoretical probability is the ratio of target outcomes to all possible outcomes.
- Empirical probability relies on analysis of observed outcomes of many repetitions of the random process generating the event. It is the ratio of all target outcomes observed to all outcomes observed.
- An empirical sampling distribution is the distribution of a statistic.

## Unit Overview

Probability relies on intuitions about chance events, but unfortunately, our intuitions are often misleading. This unit addresses everyday intuitions but provides opportunities for students to elaborate and revise their intuitions through investigation of the behavior of simple chance devices. Peering through the lens of the behavior of these devices, the unit develops probability—the measure of uncertainty--from two perspectives: long-term trends in outcomes of random events, also called empirical probability, and analysis of the structure of the devices producing these long-term trends, also called theoretical probability. In principle, these two forms of estimation should converge on common values.

**Days 1 and 2: Exploring Expectation**
Students are introduced to theoretical approaches to estimating probability by examining the structure of two-color spinners for which most students have firm intuitions about what they expect to happen during the course of multiple spins. When the two regions of the spinner have equal areas, most students expect that each of the two possible outcomes is equally likely. When the areas are not equal, most students anticipate unequal probabilities. Probability can be estimated by the ratio of the area of one portion of a spinner to the total area of that spinner. This is a gentle introduction to theoretical probability.

A second activity, Sneaky Pete, challenges students' notions of theoretical probability by partitioning the equal area 2-color spinner so that not all of the partitions of the same color are contiguous. Many students have difficulty reconciling their previous intuition that equal areas implies equal probability when the regions when the regions are not contiguous. The reason is subtle: Often students have not thought deeply about the assumptions that they are making about repetitions of the random event



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(each spin of the spinner). Upon reflection, many students do not believe that the chance of the spinner landing anywhere on the spinner disk is exactly the same from trial to trial. Yet this is the assumption that allows for the structural analysis of probability.

Empirical probability is introduced as another means for estimating probability. Each student records the outcomes of 10 repetitions (a sample) of the 50-50 Melissa spinner, either with a physical device or with a TinkerPlots simulation. Students record the number of red and blue outcomes and create a bar-graph of the results. Students often are surprised to find that the outcomes of 10 spins frequently do not copy the expected theoretical probability of 0.5, which would yield 5 outcomes of one color and 5 of the other color. But when individual outcomes are accumulated across the class, the empirical estimate is much closer to the students' intuitions about the theoretical estimate, reconciling theoretical and empirical estimates. This is an experience of the "law" of large numbers—larger samples are generally better predictors of populations than smaller samples when both sample the same process.

What Do You Expect? concludes with Mystery Spinner. Students design a 2-color spinner but do not reveal its structure. Other students guess its structure by looking only at the results of outcomes. They compare their estimated probabilities with those that might be expected from the structure of the spinner. In this activity, it is fruitful to consider the number of repetitions--the sample size--that may be needed for a good estimate.

**Day 3: Investigating Sampling**
Students are introduced to an empirical sampling distribution. They investigate the effects of sample size (here the number of trials in an experiment) on sample-to-sample variability by examining 30 runs of the percent of red outcomes for an equal probability spinner (sample size 10, 100, and 1,000).

**Day 4: Investigating Compound Probability**
Students learn to employ a sample space to generate a theoretical estimate of probability for a slightly more complex random process. Students explore the behavior of a compound event—two spinner sums.

**Day 5: Modeling Chance**
Students begin to explore probability models of events in the world. This is a theme developed more extensively in Unit 6.

**Days 6 and 7: Formative Assessment**

Students respond to a brief quiz. Student responses that represent different ways of thinking according to the Chance and Modeling Variability construct maps are deliberately compared and contrasted.

## Read

- ☐ **Unit 5**
  Start by reading the unit to learn the content and become familiar with the activities.
- ☐ **Mathematical Background**
  Reread the Mathematical Background carefully to help you think about the important characteristics of probability and chance.
- ☐ **Chance construct map**
  The concepts in this unit and possible ways students might think about them are described in the Chance Construct Map. Read the construct map and/or visit the website (modelingdata.org) to view a progression of student thinking about chance, beginning with notions of individual control of random events and progressing to include reasoning about probability by analysis of the structure of the process generating the uncertainty or by analysis of long-term structure revealed by accumulating repetitions of the outcomes of that process.
- ☐ **Sample student thinking**
  Reviewing the Chance construct map will help you to anticipate and prepare for the types of ideas your class might generate. It will also help you to think carefully about productive ways to talk about probability and chance during instruction.

## Gather

- ☐ Student worksheets
- ☐ Spinners
  Either purchase blank spinners that students can partition or use existing 2- and 3-region color spinners.
- ☐ Large spinner to display for entire class to use OR projection of computer screen with TinkerPlots.
- ☐ TinkerPlots
  An alternative to using spinners is to use the Sampler feature of TinkerPlots.

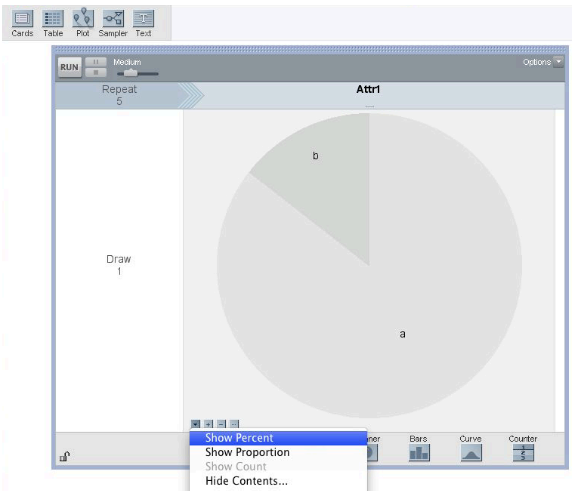
## Prepare

- ☐ **TinkerPlots**
  Although many of the tasks in the lesson can be conducted with physical spinners, some require the use of *TinkerPlots.* For example,

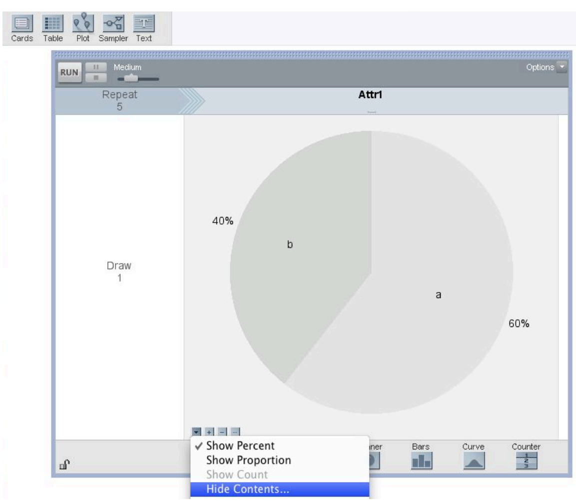
 

one task asks students to spin a spinner 1,000 times. While this is unreasonable with a physical spinner, *TinkerPlots* allows you to conduct this task in a few seconds. Spend some time reviewing the *TinkerPlots* support materials and trying the tasks out for yourself.

- ☐ **TinkerPlots Mystery spinner (if TinkerPlots is available)**
  Design a spinner with two outcome spaces. Click the upside-down triangle under the spinner as it is shown in Figure 1. Turn on "Show Percent of Show Proportion" to enter precise proportions.




Then click "Hide Contents. . ." to hide the spinner you just created. (The spinner can be 50-50 or something else, like 80-20.)




If you click the option, it will bring up the "Hide Device Contents" window. Choose "Hide Contents." The spinner will turn into a mystery spinner so that other students cannot see the design of the spinner.

## Chance and Probability

Often, uncertain events (e.g. rolling a die) have multiple possible outcomes. Any of these outcomes could happen by chance. Probability is a measure of uncertainty. It ranges from 0 to 1 in value; with zero indicating a particular outcome is impossible, and 1 indicating that the outcome is certain. Two complementary approaches can be used to infer the probability of an outcome. One, theoretical probability, relies on analysis of the structure of a chance process. The second, empirical probability, relies on observation of the outcomes obtained from repetitions of the chance process.

## Theoretical Probability

Theoretical probability is a measure of the likelihood of an event based on a sample space of known outcomes. In order to calculate theoretical probability, a full sample space of possible outcomes must be known. Examining the structure of a chance device often can help to generate a sample space. For example, when rolling a die one can determine the sample space by examining the structure of the die (i.e. 1, 2, 3, 4, 5, or 6), and probability can be calculated to measure the chance of rolling a 4 (1/6). This ratio describes the relative proportion of target outcomes that is expected in a set of repeated trials (experiment).

## Empirical (Experimental) Probability

Another method for estimating the probability of an event is empirical probability. Empirical probability is a measure of the likelihood of an event based on the results of repeated trials. A set of repeated trials can be called a sample. Each of the trials in the sample are *independent*, which means that one result does not influence the result of the next trial and that the probabilities of the outcomes are the same each time the trial is run. Although the empirical results of a sample might differ from the results expected from the theoretical probability, given enough trials the two should converge. For example, although the theoretical probability of an outcome of 1 is 1/6 for a 6-sided die, a short run of 12 tosses might result in 4 outcomes of 1, for an estimated probability of 1/3. However, the number of outcomes of 1 for a much larger number of tosses will be much closer to 1/6. The essential idea is that although the result of each trial cannot be known in advance, we can be very confident about the structure of the likelihoods of the outcomes if we can observe many repetitions of the chance process.

## Sampling Distribution

Just as whether or not we observe a particular outcome varies from trial-to-trial, so too does the value of a statistic vary from sample to sample. Recall that a sample consists of a set of trials of the repeated process. Although there are theoretical approaches to estimating the nature of the distribution of a statistic over repeated sampling, we focus solely on empirical approximations to the sampling distribution—the distribution of the values of a statistic from sample to sample. An empirical sampling distribution is obtained by simulating the results of repeated samples. The distribution of each sample is described by one or more statistics (recall that a statistic measures characteristics of a distribution) and the collection of these sample statistics constitutes a sampling distribution. For example, if we wanted to describe the distribution of the proportion of red in a 2-color, red-green spinner, we could create a sample composed of 10 trials and then calculate the proportion of red outcomes. Proportion of red will serve as the statistic of interest. We would then collect the results of a large number of repetitions of the same process (e.g., 400 samples, each consisting of 10 trials). The resulting collection of statistics, each measuring the proportion red in a sample of size 10, would show the nature of the sample-to-sample variability in the proportion red. The center of the sampling distribution will be close to the theoretical proportion, and the variability of the sampling distribution will depend on the size of each sample. Larger samples will have less sampling variability than will smaller samples of the same process.

## Exploring Expectation



Exploring Expectation consists of three activities—Melissa's Spinners, Sneaky Pete, and Mystery Spinner—that together provide opportunities to explore and to coordinate theoretical and empirical probability in contexts of simple random devices: two-color spinners. Melissa's Spinners introduces students to theoretical probability by asking them to choose among three spinners to decide which has the highest chance of landing on red. Students are then challenged to identify the spinner that they anticipate will land on red approximately ½ of the time, imagining repeated spins. Probability of red is introduced as the ratio of the red area to the total area of the spinner.

The Sneaky Pete activity revisits probability as the ratio of the area of red to the total area but does so for noncontiguous areas of red. This lack of contiguity often unmasks hidden assumptions. Each time we spin the needle, we assume that it is equally likely that the needle will land on any part of the spinner. But, some students will suggest that different partitions lead to different expectations. Hence, Sneaky Pete provides a way of thinking about probability as a ratio of the area of the target outcome (also called an event) to the area of all the outcomes. It assumes that in every trial (repetition), the probability of red does not change.

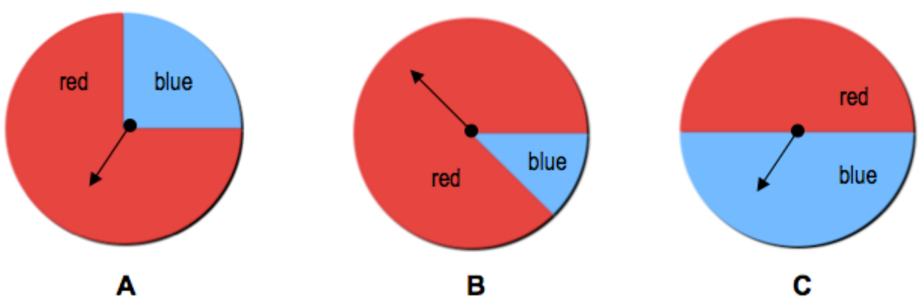
After these explorations of theoretical probability, students explore the behavior of Spinner C (the equal probabilty 2-color spinner in Melissa's Spinners). After thinking about Sneaky Pete's spinner the students experiment with Melissa's spinner. Each student records the outcomes of 10 repetitions (a sample) of Spinner C, either with a physical device or with a TinkerPlots simulation. The first surprise for many students is that the outcomes of 10 spins often are not copies of the theoretical probability of 0.5, which would yield 5 outcomes of one color and 5 of the other color. But when individual outcomes are accumulated across the class, the empirical estimate is much closer to the theoretical estimate. This is an experience of the "law" of large numbers—larger samples are generally better predictors of populations than smaller samples when both sample the same process. And, in the long run, theoretical and empirical estimates should converge.

Last, students design their own Mystery Spinner. Other students try to guess its structure using only outcomes, because the structure of the spinner is hidden.

## Whole Group

**1. How does the design of a spinner influence the chance that the needle will land on a color?**

a. Distribute the 'Exploring Expectation: Melissa's Spinners' worksheet to each student.




b. Tell students that Melissa is thinking about what would happen if she spun the spinners again and again. Support students in predicting the outcome of a set of trials, using questions such as these:

Q: For each spin of A, what could happen? How about for B? C?
Q: If each person spins each spinner 10 times, which spinner will likely land most often on red? Why do you think so?

*Note:* Documenting student thinking on the board will help this conversation.

c. Ask students to talk about which spinner Melissa should choose if she wants it to land on red approximately half the time and blue approximately half the time. Ask students what about the spinner they chose makes them confident that it will land on red about half the time and blue about half the time.

d. Define probability as the ratio of target outcomes (in this case either red or blue) to all outcomes (red and blue) and tell that probability is a measure of chance. Show students that spinner C has one blue section and one red section (two sections total) and that both are the same size. This can be used to determine that the probability of getting blue is ½ and the probability of getting red is ½. Ask:

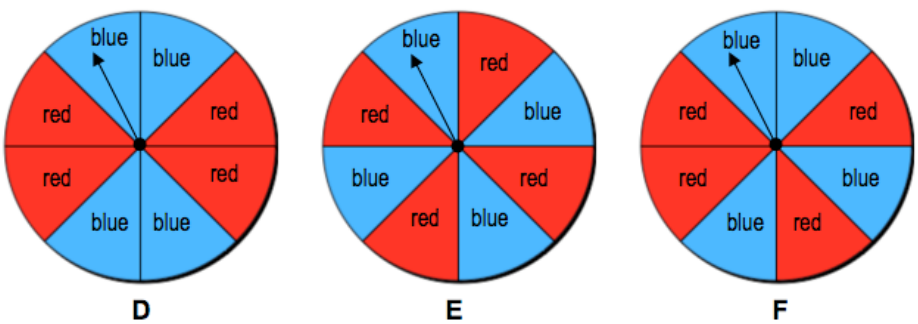
**Construct: Cha1, Cha2, Cha3**
This task engages students in thinking at the early levels of the Chance construct.

**Construct: Chance**
See the Chance construct for a description of sample space.

a. Q: What is the probability of red for spinners A (3/4) and B (7/8).
b. Q: Is probability a good measure of uncertainty? Why do you think so?

**2. Elicit students' expectations of Sneaky Pete's spinners.**

a. Distribute the 'Sneaky Pete' worksheet to students.
b. Tell the students that Sneaky Pete made 3 spinners and said that




they would all give about the same results, even though they look different. Have students think about whether they agree with Sneaky Pete, using questions such as these:

Q: Do you agree with Sneaky Pete? Why?
Q: What do you think will happen when Sneaky Pete spins each one, again and again?

*Note.* The aim of these questions is to develop and probe students' understanding of theoretical probability, as determined by the ratio of areas or the ratio of outcomes. For example, spinner D has 8 congruent sectors, four of which are blue and four of which are red. The probability of landing on red is obtained as [4 red/(4 red + 4 blue)]. This is an opportunity to think about sample space as a tool for measuring chance. If students need further convincing, cut out the sectors of any of the spinners (D, E, F) and re-arrange them to establish that the ratio of red sectors to (red and blue) sectors is ½.

Exploring Expectation
Investigating Sampling
Investigating Compound Probability
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c. Have students think about the possible values for the probability ratio and their meaning, using questions such as these:

Q: What is the highest value that the probability of an event occurring can be? Why?
Q: What would a spinner look like if the probability of red were 1?
Q: What is the lowest value that the probability of an event occurring can be? What would that mean?
Q: What would a spinner look like if the probability of red were 0?

## Individual

**3. Provide an opportunity for students to try out the 50-50 Melissa spinner.**

During this activity students are introduced to empirical probability and explore its relationship with theoretical probability.

a. Give each student a spinner that has equal red and blue regions (Melissa's spinner "C") and a worksheet to record the results of their spins.

b. Tell the class that each student will spin their spinner 10 times. This is called a 10 trial sample. A sample represents a process. A process occurs an infinite number of times (this is called a population), so we can only look at some finite number of these times. Tell them to record the result of each trial. Emphasize that the 10 trials is a sample of size 10.

c. Give students a few minutes to spin their spinners and to record their results. Use the worksheet and have each student construct a bar graph of the outcomes. Display the graphs on the wall of the classroom. Invite students to look at them and to think of a way to organize the graphs so that a trend or pattern might be more noticeable. Ask students to share what their ideas about organizing the individual graphs to make patterns more noticeable.

*Note*: It is often striking when the graphs are clustered so that those with 5 red, 5 blue occupy the center and the remaining graphs are arranged to the left (e.g., 4 red, 6 blue) and right (e.g., 6 red, 4 blue) of this middle clump.

d. Looking at the displays reorganized to exhibit a center clump, ask students to visually estimate the center of the distribution displayed.

*Note.* The goal here is to start to build the intuition that there is always sample-to-sample variability. Moreover, small numbers or batches of trials may not look much like the theoretical relative frequency, but longer runs tend to converge to it. Here, if all the spins are summed and the ratio of the number of red outcomes (r) is compared to the total number of outcomes (r +b), the probability tends toward 1/2. Inviting students to reorganize the display should make the central tendency of the estimated probabilities of the collection of samples more visible.

## Whole Group

**4. Discuss variability in students' results and support students to think about combining their results.**

*Note.* Consider the notion of a trial – an aspect of a phenomenon that is considered repeatable. What are we assuming when we treat each spin of the spinners as equivalent? It is this attribution of "sameness" that allows us to combine results obtained by different experimenters (people spinning literal spinners) or the cybernetic spinners from a different computer. The probabilities of each set of outcomes (e.g., 10R, 0B; 9R, 1B, etc.) is displayed in the Appendix, along with a discussion about how these probabilities can be determined in principle.

a. Support students to think about why trial results vary, asking questions such as:

   Q: Why did many of us have different results?
   Q: Did everyone get the results they expected?
   Q: Now that everyone has spun spinner C ten times and recorded the number of reds and blues, can we combine everyone's results? Why or why not?

b. Add trial results to find the total number of times (across all spins conducted by all students) the spinner landed on blue and the total for red. Support students in thinking about the advantage of larger numbers of spins, using questions such as these:

Q: Which is better, to look at just one person's result or to look at everyone's results? Why?
Q: Which is closer to what we expect—this person's results (teacher chooses an extreme example) or the combined results? Why do you think so?
Q: Do you think this person influenced the spinner somehow (pick an extreme result)? How could this happen?
Q: What might happen if we did this again?

c. Use all the trials to estimate the probability of a red outcome. Ask students: how does this estimate compare to the visual estimate?

## Small Group (if Tinkerplots is not available)

**5A. Have students design and predict the structure of mystery spinners *without* TinkerPlots.** Distribute the worksheet, 'My Mystery Spinner (Blank Spinner).'

a. Give students a blank spinner and tell them to design a secret two-color spinner.

*Note.* Students will need to know how much of the area is covered by each color. For example, one might make 1/4 of the area red and the rest black. The smallest amount of area that should be covered by one color is 1/10 of the total area. It is important that students work with fractions that are both known and large enough to be realized when the spinner is spun so that students can investigate the relationship between the structure of the device and the empirical data.

If you have access to TinkerPlots, then the activity changes to focus more on sample-to-sample variability. Using TinkerPlots capabilities, students can construct mystery spinners in the sampler and run them many more times. See the optional activity at the end of this unit for details.

b. Tell students to spin their spinner 40 times and record the results. They should then hide their spinner and share the results with a partner. The partner draws what he or she thinks the hidden spinner looks like.

c. Have students consider the accuracy of predictions based on the results of 40 spins. As needed, you might ask questions such as:

Q: Was your partner right?
Q: How close did he or she come?
Q: What strategy did your partner use to figure out what your spinner might be?
Q: Did you use the same strategy or a different one?

## Whole Group (if TinkerPlots is available)

**5B. Have students design and predict the structure of mystery spinners *with* TinkerPlots.** Distribute the worksheet 'My Mystery Spinner.

a. Show the class the Sampler in TinkerPlots and demonstrate how to create a two-region, equal probability spinner. Then show the class how to hide the spinner.

b. Ask the class how someone might be able to guess about the structure of the spinner if it were hidden. Set the number of repetitions to 10 and the draw to 1. Then run the spinner and demonstrate how to graph the outcomes and to find the percentage of each color. Ask the class what they might guess about the structure of the spinner just by looking at the graph. Ask the class what might happen if the experiment was run again (This is a good informal assessment of what they have learned.)

## Pairs

c. Working in pairs, one student designs a 2-color spinner and hides it. No color can occupy less than 10% of the area of the spinner. The other student first uses 10 repetitions and then as many as he or she likes to guess at the structure of the spinner. After both guesses, the structure of the spinner is revealed.

## Whole Group

d. Prepare in advance, a TinkerPlots Mystery Spinner, with 2 colors, red-blue, with 5% of the area blue and the rest, red. During a whole-group discussion, ask:

Q: How did the guess with 10 repetitions and more than 10 compare? Why did that happen?
Q: Here is a mystery spinner (5% area blue and the rest red). I'll run it 10 times. Guess its structure. Now I am going to run it 100 times. Guess its structure. Here's the structure. Why were 100 repetitions (remind students that we call this a sample) better than 10? (*Note:* if it is not obvious that 100 is better than 10, repeat the experiment)

## Students' Ways of Thinking: Expectation

Exploring Expectation explores theoretical and empirical probabilities. For ten repetitions of spinner C, most students predict mild variation (between 4-6) around an expected value of 5 red (or 5 blue), but a few predict no variability. For example, after finding the sum of red and blue outcomes for the entire class, a fifth-grade student suggested: "I think that 55% red (of all spins made by all students of a 50-50 red/blue spinner) is surprising, because the spinner is half red and half blue." This student seemed to believe that a theoretical probability is copied in the sample. Even students who predict variability are often surprised by its extent, often attributing variability among the results obtained by each student to determinate causes, such as "maybe some spun it too slow."

Other students will suggest that variability is inevitable, but that some events are more likely than others: "I don't think it is really surprising because results like 1/9, 4/6, 6/4, 3/7, those results throw (it) off course. Cause it is more likely to get 5/5 (red/blue) than 1/9. Only a person got it. The results have more chance to be 5/5. I am not surprised." This student appeared to be suggesting that although one student's results were one red and nine blue spins (1/9), this result was not as likely as those around 5/5.

Students will often notice that larger samples give better estimates of Mystery Spinners but not be able to explain why. Sometimes, it is helpful to use extreme cases—It is hard to detect a low frequency, improbable event without many repetitions. Most students grasp this readily when it is pointed out.

It is more difficult, but critical, to talk about why an outcome such as 9R 1B is more likely to appear in 10 trials than is 90R 10B for the same equal probability two color spinner. Many students think that if the ratio is the same (9:1), then no matter the number of trials, one is as likely to see 9r, 1b as one is likely to see 90r 10b. One way to provoke some further reflection is to build on intuitions about how unlikely a long string of 90R would be, compared to 9R, for each spin of the 50-50 2-region device (or for each toss of a coin). The aim is not to develop the formal probability analysis (see Appendix), but rather to develop the intuition that unlikely events are more apt to occur with smaller samples—smaller samples are more variable. We will aim to develop this more systematically in the activities devoted to the sampling distribution.

## Investigating Sampling



Investigating Sampling introduces the concept of a sampling distribution with an equal probability, two-color spinner in the now-familiar context of samples of 10 spins. Students first examine sample-to-sample variability, then decide on a measure for each sample, a statistic that they want to keep track of, and conclude by using TinkerPlots to construct a sampling distribution consisting of 300 samples. The aim is to highlight decreasing sample-to-sample variability with increasing sample size. In order for students to notice the decreased variability it is important to use the data display and statistics concepts from units 1-4.

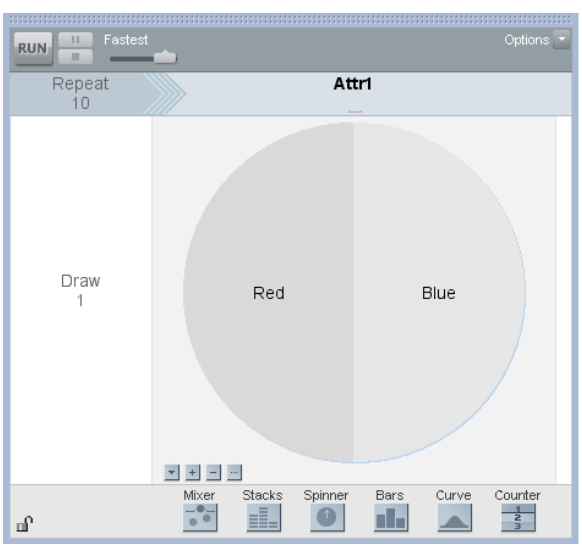
### Individual or Small Group

Construct: Cha3 and Cha4

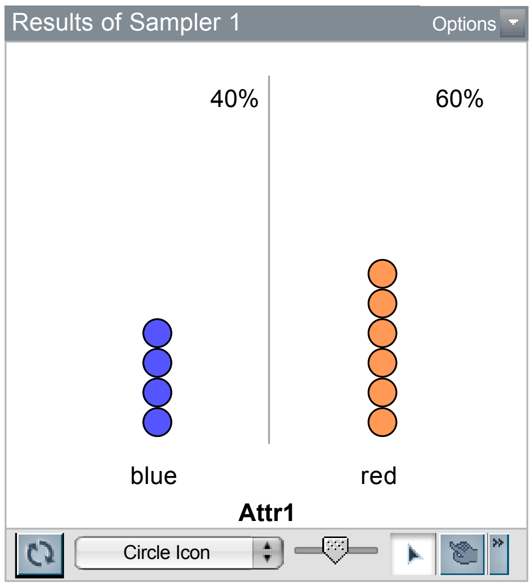
**1. Construct sampling distribution for samples of size 10.**

a. Remind students that when they see a spinner that is ½ red and ½ blue they now know that if spun again and again, they can expect the pointer to land on red ½ of the time and on blue ½ of the time. But also remind the class that if we spin some particular number of times, results will vary: in some cases the spinner will land on blue more often, in others red will be more common.

b. Distribute the worksheet, 'Investigating Sampling.' Ask students to construct a TinkerPlots sampler for a 2-color, red-blue spinner with 10 trials. Let students work in pairs but check to be certain that each student can create the simulation independently.



c. Have students run the sampler and plot the outcomes for 10 spins. Use TinkerPlots to show the percent blue and the percent red. Ask students: What does the percent blue mean? (How many of the 10 spins were blue?) The percent red? (How many of the 10 spins were red?)




d. Ask students to run the sampler a few more times and to record an observation about what happens and why they think it happens that way. Focusing on percent blue, what did they notice? (They should notice that it changes, especially across the entire class.)

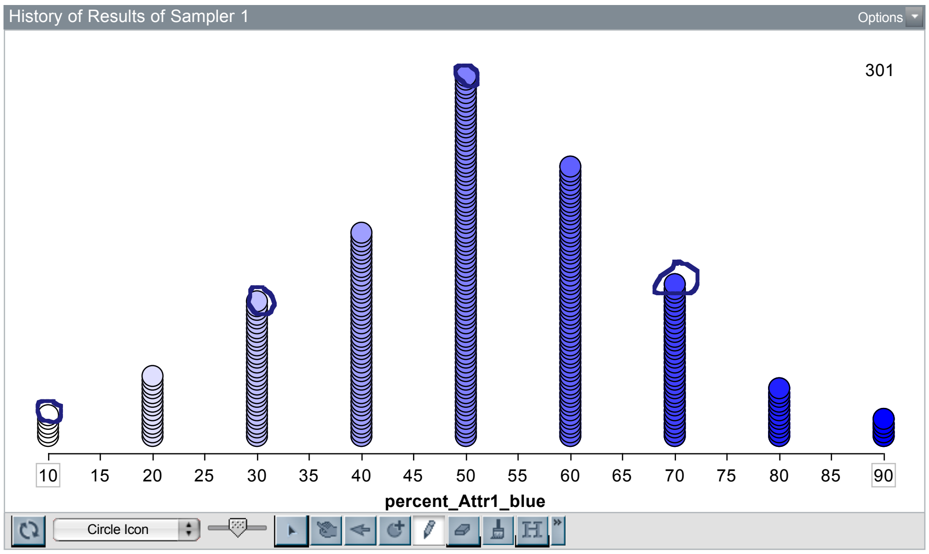
e. Ask students to collect 300 samples of size 10 (the 10 trials) instead of just a few. Before they do so, ask questions such as these:

Q: If we collect 10 trials at a time, and do this 300 times, what do you predict the lowest percentage of blue outcomes will be? The highest? Where do you expect most of the percent blue to fall? (from X% to Y%)

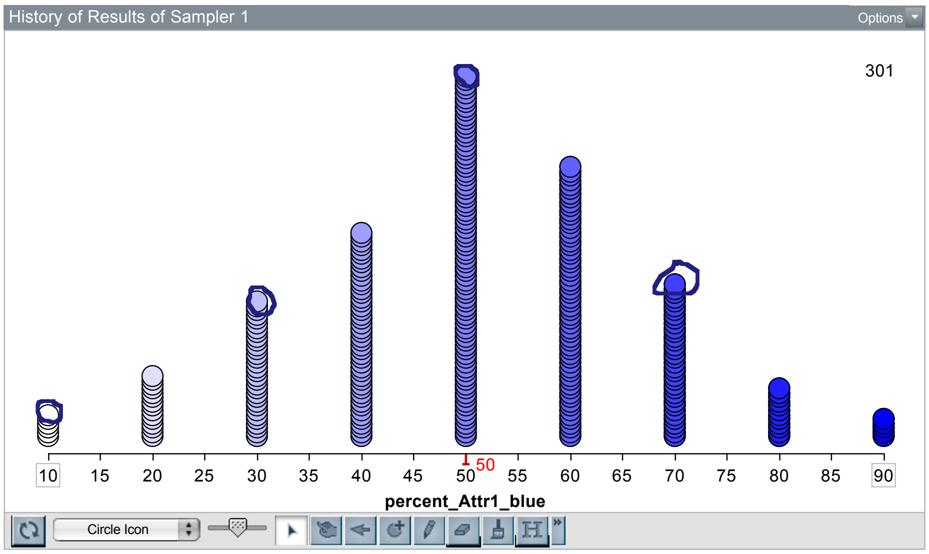
f. Ask students to plot the result of the 300 samples they have collected. To help students interpret what they are seeing, ask questions such as these:

Q: Let's look at a few of the cases on this display of the sampling distribution. What does each case represent? (the percent blue for a sample of 10 trials or spins) Point to a few different cases, as shown in the figure below, to make certain that students know that each case represents ONE sample and that there are 300 samples represented in the sampling distribution.



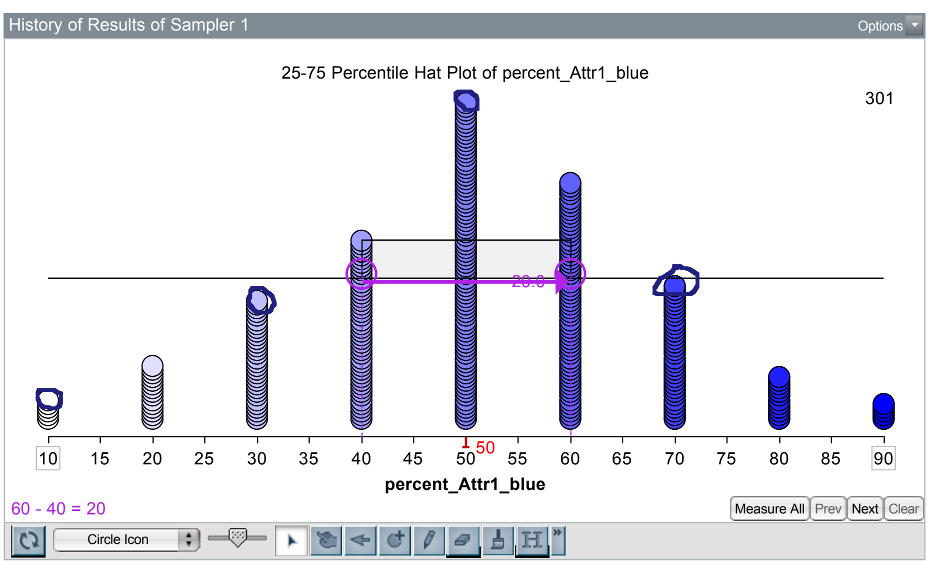



g. Ask students how they might measure the central tendency of the percent blue across all the samples. What might the median of the sampling distribution tell us? (In the figure below, the median is shown, and it estimates the percent blue in the spinner used to produce each sample). Help students notice that samples range from low to high percentages of blue, but the central tendency of the sampling distribution is a good estimate of the real percent blue in the spinner.

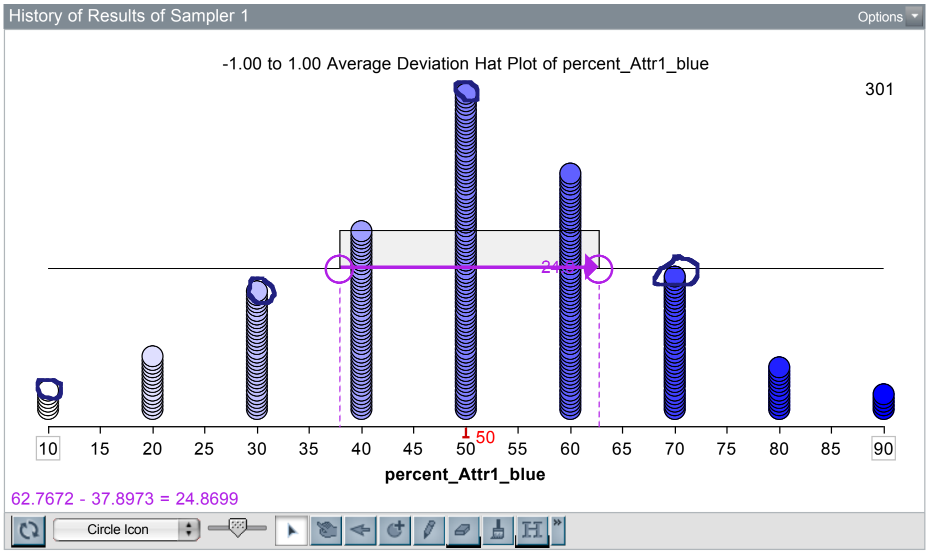



h. Ask students to come up with ways to describe about how much the percent blue tends to change from sample-to-sample.

Note: In the sampling distribution shown below, the IQR measures the tendency of the percent blue to vary from sample-to-sample, suggesting that the mid-50 of percent blue ranges from 40 to 60 percent. The average deviation plot suggests that on average, a sample's percent blue tends to be about 12 ½ percent above or below the mean percent blue (the true value of 50% blue is estimated as 50.3 by the mean).

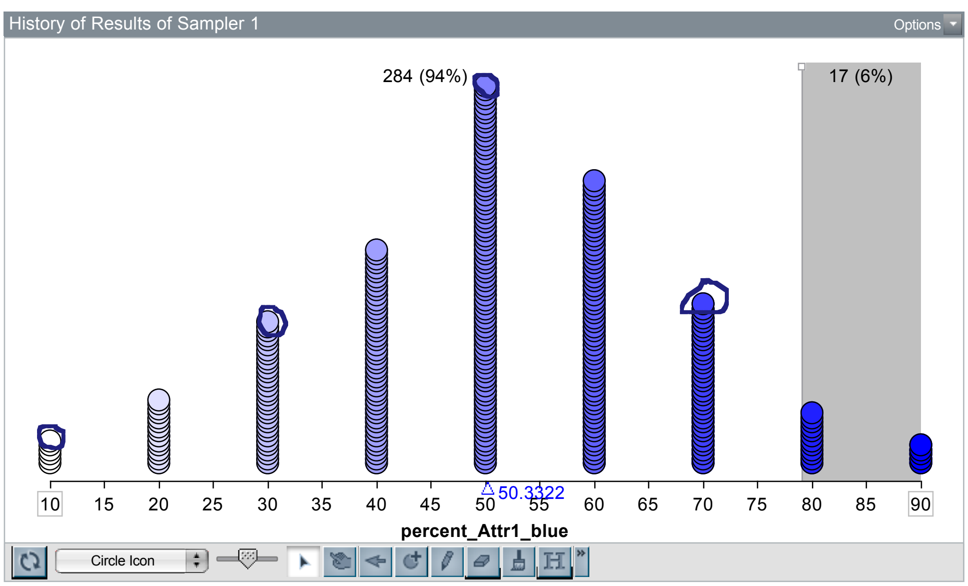


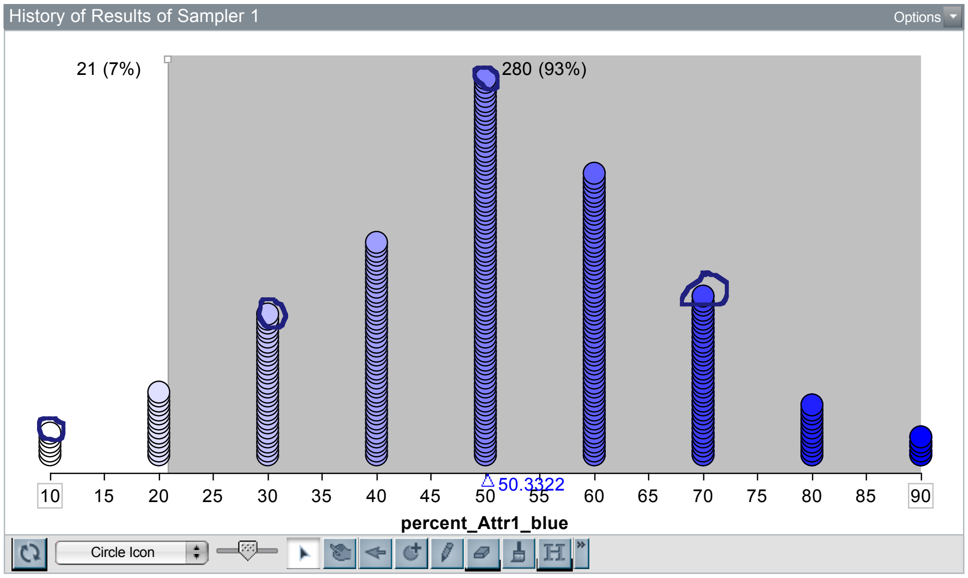



i. Ask students to estimate the likelihood that with a sample size of 10, the percent blue in any particular sample will be 20 percent or less. Ask students to estimate the likelihood that with a sample size of 10, the percent blue in any particular sample will be 80 percent or more.

*Note:* The Divider tool and Percent tools can be used to develop these estimates as shown below.






## Individual

**2. Construct sampling distributions for samples of size 100, 1000.**

a. Ask students to change the number of repetitions to 100 and to collect 300 samples the same way they did with samples of size 10. Before students begin to conduct the simulation, ask them to predict:

Q: If we collect 100 trials at a time (we spin the spinner 100 times and record the number of blue and red outcomes) to have samples of size 100, what do you predict the lowest percentage of blue outcomes will be? The highest? Why do you think so?

Q: Where do you expect most of the percent blue to fall? (from ____% to ____%)

b. Have students run the simulation and plot the sampling distribution. Remind them to use the same scale for percent blue as they did earlier.

*Note:* Students often forget to worry about scale. So, they sometimes mistakenly conclude that sample size does not matter.

c. Have students repeat the investigation for samples of size 1000.



## Whole Group

**3. Lead a discussion about what changes in the experiments with larger numbers of trials.**

a. Support students in thinking about how their results change as more repetitions are used. As needed, ask questions such as these:

Q: What do you notice?
Q: What stays the same?
Q: What changes?
Q: Why do you think this happens?

b. Have students use one of the measures of precision (or variability) developed in unit 3 (for example, IQR or Average Deviation). Tell them to use the measure to compare the variability of the percentage of blue from the three different sample sizes.

c. Assess what students have noticed about the variability of results in the different sample sizes. As needed, use questions such as:

Q: What do you notice about how the percent blue changed from sample-to-sample with sample sizes of 10, 100, and 1000? Why did the median of the sampling distribution stay about the same for all three sample sizes?

Q: Thinking about how the median changes from sample-to-sample, what is the advantage of larger samples?

### Students' Ways of Thinking about Sampling Variability

When posing the task, the teacher asked students to predict what might happen. All suggested, perhaps based on their previous experiences, that there would be sample-to-sample variability.

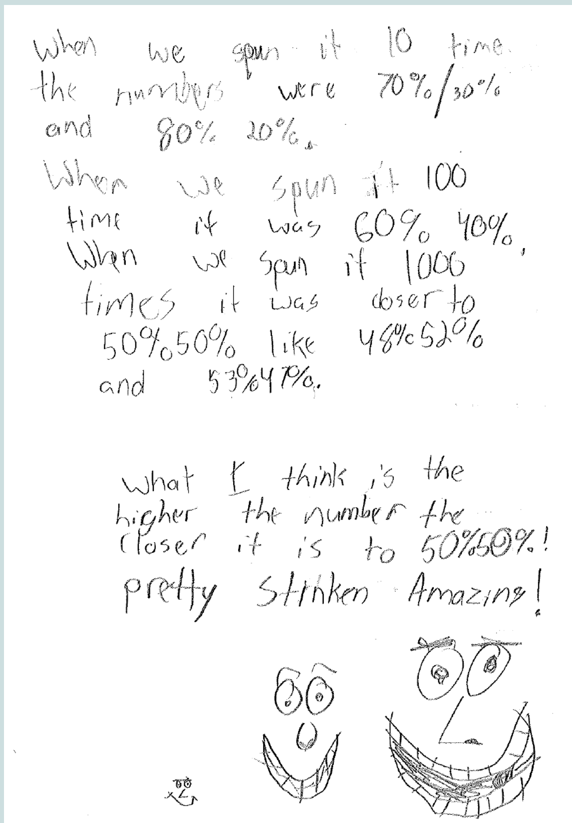
However, some students thought that more repetitions would increase, not decrease, the percentage of unusual proportions, such as 8 red, 2 blue. They reasoned that if one could see outcomes such as 8 red and 2 blue in 10 spins, then there would be even "more chance" of such a result in more spins (such as 100 or 1,000). Hence, seeing and tracking the results of different sample sizes (10, 100, and 1,000) was an important source of information about the behavior of a repeated process.

After observing this variability, students often reported intuitions of events "balancing out" resulting in less variability for more repetitions of the process (e.g., 10 vs. 100 vs. 1,000). For example, one student suggested that if a sequence of 10 spins resulted in an unusual outcome such as 8R and 2B, this might be balanced in the long run by another unlikely string, such as 2R, 8B or 3R, 7B that occurs as the number of spins is increased.

Although chance does not act via a balance mechanism (e.g., the gambler's fallacy is that if one is having a string of bad luck, perhaps one should increase the bet because your luck is bound to change), the intuition expressed by this student is sensitive to reasoning about a long-term process. And, the student did not believe that they could predict when the balancing string might appear—only that if the process kept running, it would eventually happen. (The difficulty with this explanation is that it is equally likely that if the process kept running, imbalance might increase (e.g., another 8R, 2B is as probable as a 2B, 8R).

Students are helped to think about chance and long-term process by keeping track of what is staying the same (e.g., the area devoted to each color, the strike of the pointer, etc.) and what is changing (the number/percent of red from sample to sample).

Here are some examples of students' thinking when they compared the variability of 300 runs of a two-color (1/2 red, ½ blue) spinner, with an eye toward tracking the percent of red (or blue) when the number of spins was 10, 100, or 1000. The student work suggests that they noticed that smaller sample sizes (10 spins) resulted in more variability when compared to larger sample sizes (100, 1000).



When we spun it 10 time the numbers were 70%/30% and 80% 20%.
When we spun it 100 time it was 60% 40%,
When we spun it 1000 times it was closer to 50% 50% like 48% 52% and 53% 47%.

what I think is the higher the number the closer it is to 50%50%!
pretty Stinken Amazing!

## Investigating Compound Probability

Students explore the behavior of a compound event—two spinner sums. They enumerate the different ways in which each sum can be formed (permutations). While working to enumerate the sums students have the opportunity to consider the need for order. For example, is spinning a 3 and then a 1 the same as spinning a 1 and then a 3? Running simulations will show students that all permutations are needed to describe a full sample space. Students also use this sample space to predict what will happen over 180 trials. The class pools its data and compares the prediction to empirical outcomes.

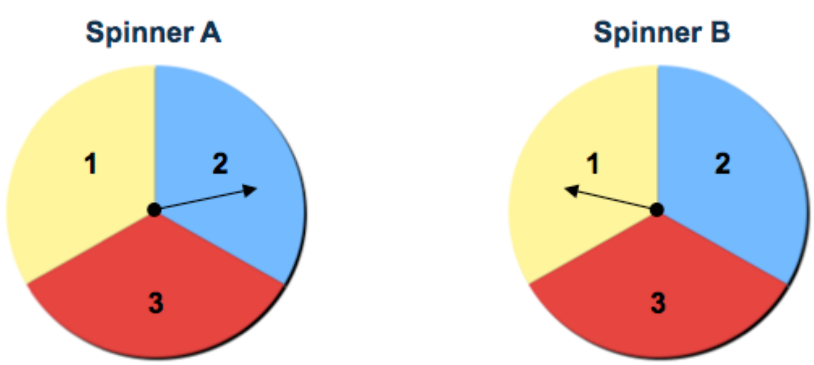


Construct: Cha5 and Cha6

### Whole Group

**1. Introduce Investigating Compound Probability.**

a. Present the class with the image of two spinners and hand out the 'Investigating Compound Probability' worksheets.




b. Explain to the class that both spinners will be spun at the same time and that the two numbers the arrows are pointing at will be added together.

c. Using the two spinners, conduct one or two trials to make sure students understand what it means to combine two spinners into one event (a compound event). In this case the outcome of each trial is the sum of the results of the two spinners.

d. Help students think about possible and impossible outcomes. As needed, use questions such as these:

Q: If we spin the spinners at the same time and add the values that appear on each spinner, what are the possible outcomes (or sums)?
Q: Which sums are not possible?

*Note.* Students might begin to notice ideas about considering order in the sample space early in the lesson. If they notice this, then feel free to make these ideas clear throughout the lesson. However, if no one noticed this earlier, then the next few activities will make this clear for students.

2. **Support students in developing the idea of sample space, or all the possible pairs of results from the spinners.**

   a. Help students to think about the different ways that particular sums could happen. As needed, ask questions such as these:

      Q: What are all the possible pairs of results that could happen when we spin the two spinners together?
      Q: To help you get started, think about what might happen if the pointer on spinner A landed on 1. What might the pointer of spinner B land on?
      Q: What would the sums be?
      Q: If spinner B landed on 1, what might the pointer on spinner A be on?
      Q: What would the sums be?

   b. Tell students to list all the ways that each sum could be formed on the worksheet. Use questions like the following as needed to help students think about their sample space:

      Q: Is a 2,1 result the same as a 1,2 result?
      Q: Do any of the sums have more than one way to occur? Does that matter?

      *Note.* Make note of the ways in which student go about making the possible sums. For example, do they treat a 1 on spinner A paired with a 2 on spinner B (1,2) as different from a 2 on spinner A paired with a 1 on spinner B (2,1)? Some students treat these as identical.

   c. Help students see the relationship between the possible pairs that make each sum and the likelihood of each sum, by asking:

Q: If we spin them again and again, which sum will be most likely, or will all the sums be equally likely? Why?
Q: Does it matter that some outcomes have more than one possible way of occurring? Does this have anything to do with how likely each outcome is?

## Small Group or Individual

**3. Have students predict the number of times each sum would occur.**

a. Tell students to use their sample space (be sure to call this the sample space) to predict the number of times they would expect to see each sum if they spun both spinners 180 times. Have students record their predictions in the table on the worksheet.

b. Remind students that the ratio of the number of target outcomes to the number of all outcomes is called probability. Tell students to use their predictions to determine the probability of getting a sum of 4.

**4. Tell students to spin the spinners.**

Tell students to run the spinners 180 times (if you are using TinkerPlots; if not, have each student try it about 20 times and combine results across students). Have students record their results in the table on the worksheet ('Investigating Compound Probability').

**5. Support students to compare predictions with observed outcomes.**

a. Support students in comparing their predictions with the observed outcomes in order to highlight the need for a sample space consisting of all permutations. Use questions such as these as needed:

Q: Considering the probability of the sum of 4 again, what do your results suggest that it should be?
Q: What do you think accounts for the difference?
Q: Do your results suggest that the sample space should include different orders to get a sum (rolling a 2 then 1 opposed to rolling a 1 then 2)?
Q: What ways of getting a 4 should we have listed in our sample space?

*Note.* Students often do not list all possible outcomes in the sample space and as a result, do not accurately predict the relative frequencies of the sums. You might need to focus attention again on the sample space and the use of permutation as a means for enumerating the sample space. The following table might prove useful:

| If Spinner A lands on | Spinner B could land on | Outcome (A,B) | SUM |
|---|---|---|---|
| 1 | 1 | (1,1) | 2 |
| 1 | 2 | (1,2) | 3 |
| 1 | 3 | (1,3) | 4 |
| 2 | 1 | (2,1) | 3 |
| 2 | 2 | (2,2) | 4 |
| 2 | 3 | (2,3) | 5 |
| 3 | 1 | (3,1) | 4 |
| 3 | 2 | (3,2) | 5 |
| 3 | 3 | (3,3) | 6 |

*Note.* There are nine different outcomes possible when one spins A and B. For example, spinner A could land on 1 and B on 1. Sums of 2 or 6 can only occur if the spinners both land on 1 or both land on 3. The probability of a sum of 2 is 1/9 and so is the probability of a sum of 6. Sums of 3, 4, and 5 are more likely. But the probability of a sum of 4 is 3/9 (because it can result from 3 different outcomes) and the probability of a sum of 3 is 2/9, as is the probability of a sum of 5 (2/9). The sum of the probabilities is 1.

b. If you are using TinkerPlots, ask students to predict what would happen in the next several trials of 180 spins. Then run the simulation and ask students what they notice. Or, increase the number of spins to a multiple of 180 and solicit predictions. Does the difference between theoretical and expected frequencies (as a proportion of the total) tend to decrease as the number of spins increases

### Students' Ways of Thinking: Compound probability

**Possible Sums.** Most students in 5th and 6th grade classrooms typically understand that 2,3,4,5, and 6 are possible sums, and that 0,1, and any numbers above six are impossible sums. However, a few students did not understand the distinction between possible and impossible sums, so it was necessary to have them literally find each possible sum using the two spinners.

**Find the Ways.** Some students suggested that if spinner A landed on 1 and spinner B landed on 2, this was the same outcome as would happen if spinner A landed on 2 and spinner B landed on 1. So they treated (1,2) and (2,1) as equivalent because the sums were all 3. That is, they focused on combinations. Hence, for these students the outcome space was: (1,1), (1,2), (1,3), (2,2), (2,3), and (3,3). Some students disagreed with this way of thinking, and suggested that (1,2) and (2,1) were different because "1 is from Spinner A and 2 is from Spinner B on (1,2) and 2 is from Spinner A and 1 is from Spinner B on (2,1)." Students said although the sums were the same, but they were different ways of getting sum of 3. These students were attending to permutations. For these students, the possible outcomes were (1,1) (1, 2), (2,1), (1, 3), (3, 1), (2,2), (2,3), (3, 2) and (3,3). To help students distinguish between these two methods of constituting the sample space, it is important to have students predict the relative frequency of each sum (see next section).

**Make a Prediction, Predicting the Results of 180 Trials.** Students typically make predictions based on one of three forms of thinking. Some students observe the equal partitions of each spinner and conclude that because these partitions suggest that the outcome of each spinner is equally likely, then the sums across two spinners are also equally likely. A student claimed, "In my opinion, all the sums will be about equally likely, because all the numbers take up an equal amount of space, since they are in thirds." These students suggest that each sum will appear approximately 36 times (180 divided by 5, the number of distinct sums). Students who attend to combinations typically suggest that the sum of 4 is most likely, because it can be formed in two ways: (2,2) and (3,1) = (1,3). Other sums can be formed in only one way, as depicted in the Table.

| Sum | Ways to make the sum | Expected total in 180 spins |
|---|---|---|
| 2 | (1,1) | 30 (1/6) |
| 3 | (1,2) = (2,1) | 30 (1/6) |
| 4 | (1,3) = (3,1) and (2,2) | 60 (2/6) |
| 5 | (2,3) = (3,2) | 30 (1/6) |
| 6 | (3,3) | 30 (1/6) |

Students who considered permutations suggested that the sum of 4 would be most likely because there were three ways of getting it. For example, a student said, "The sum of four would be most likely because it has the most ways to equal it." Another student specified how many ways to get sum of 4 by saying, "I think sum 4 will be most likely because there is 3 different ways to get it. There are only 1 or 2 ways of getting the other sums (2,3,5,6).

| Sum | Ways to make the sum | Expected total in 180 spins |
|---|---|---|
| 2 | (1,1) | 20 (1/9) |
| 3 | (1,2) (2,1) | 40 (2/9) |
| 4 | (1,3) (3,1) (2,2) | 60 (3/9) |
| 5 | (2,3) (3,2) | 40 (2/9) |
| 6 | (3,3) | 20 (1/9) |

Although students who thought about combinations and permutations both correctly predicted that 4 would be the most frequent sum, many could not calculate the expected frequency. For some students, the number of ways to make each sum was not coordinated with the total number of ways to make all the sums. For example, using combinations, one would predict a probability of 2/6 or 1/3 for the sum of 4. Many students knew that there were 2 combinations related to 4, but did not find the ratio of 2/6. Other students did find this ratio (the probability) but did not understand that it could be used to find the expected frequency: 1/3 of 180 = 60

Some students correctly predicted the theoretical expectation by creating the sample space and finding each expected probability as the ratio of the number of ways to make that sum to the total number of permutations. For example, a student predicted an expected frequency of 20 for Sum 2 and 6, 40 for Sum 3 and 5, and 60 for Sum 4. The student explained, "I did this because there are 9 possible ways to get (a sum of) 2,3,4,5,6. There is one possible way to get 2 out of those 9. So I put 1/9. 1/9 x 180 =20 spins. I did this for each number."

**Try it Out.** A 5th grade class designed two spinners with TinkerPlots and ran them 180 times.

| Sums | Predicted (of 180) | Observed (of 180) |
|---|---|---|
| 2 | 20 | 14 |
| 3 | 40 | 39 |
| 4 | 60 | 64 |
| 5 | 40 | 46 |
| 6 | 20 | 17 |

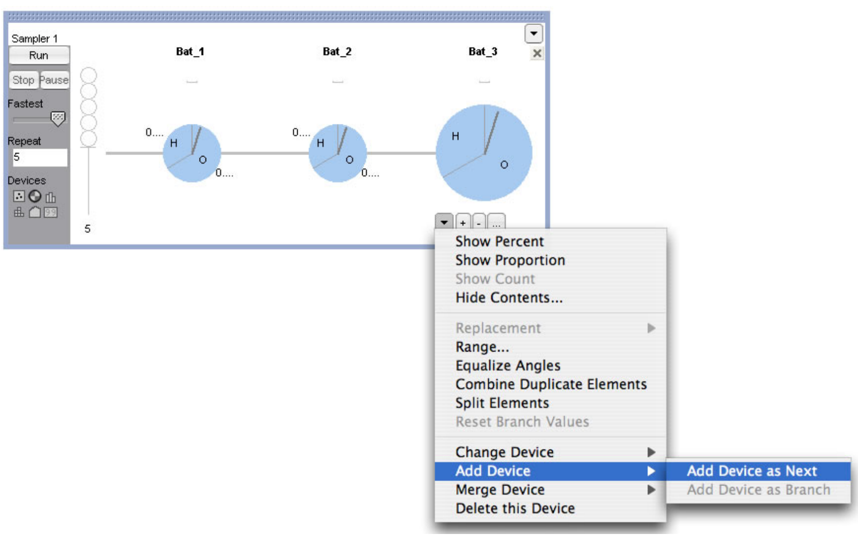
The class agreed that the observed frequencies were pretty close to the predicted frequencies for permutations but not for combinations. The teacher asked, "What do you think will happen if we just kept spinning?"

A student who had thought that all sums would be equally likely answered, “Some of the numbers that arise a little bit to fall,” which could be interpreted that the student believed that the frequencies of all sums tend to be more uniform (The teacher asked this student to run the simulation again to check his conjecture.) Another student who correctly predicted theoretical expectation said, “Because the more times you do it, more precise. It takes you closer to the right amount you should get.” This student’s intuition was that as the long-run process was repeated, the gap between expected and empirical relative frequencies would tend to be reduced, perhaps converging toward no substantial difference.

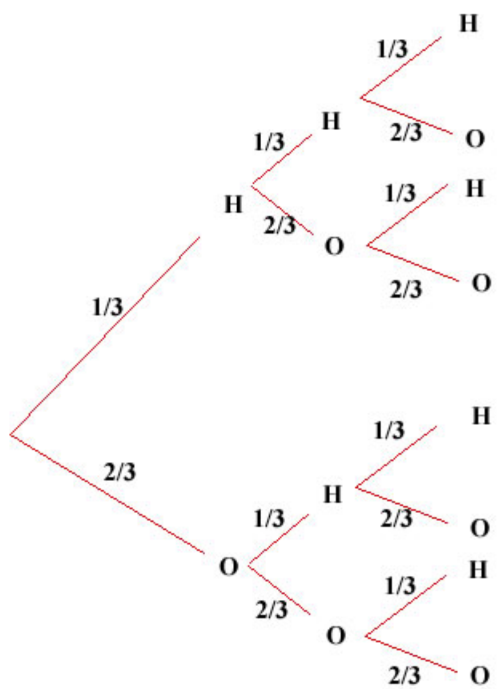
 

## Modeling Chance

In 'Game on the Line' and 'Alex Rodriguez' (see student worksheets in this unit), students make theoretical predictions for several situations involving sports. They compare their predictions to simulations conducted with spinners. In Game on the Line, students estimate the probability of three successive missed free throws in basketball, where the probability of missing is ½. In Alex Rodriguez, students estimate the probability that Alex will get three hits in his first three times at bat, if the probability of a hit each time is 1/3. Both problems can be approached either theoretically or empirically. Empirical approaches require building models. For example, the Alex Rodriguez problem can be modeled with a simulation involving three spinners as shown in the illustration below. To create the simulation, partition each spinner into sectors for the probability of a hit (1/3) and an out (2/3). You add spinners by clicking the **Add Device** button or by dragging the device to the right of the current device.




The sample space for this model does not have equally probable outcomes, so it is a bit more difficult to lay out when compared to Game on the Line. But, if we use probabilities and know that the probability of a joint event, such as HHH, is the product of the independent probabilities, we can construct a tree as follows:




We do not recommend that all students consider this more complex sample space. However, if some of your students are ready for this more complicated sample space, this is a good problem to consider.

## Formative Assessment



1. **Administer the quiz.**

2. **Use the scoring guides to score student responses.**

3. **Use Buttoned Shirts responses to generate a discussion of probability as a measure of uncertainty and how chance operates to produce a distribution of outcomes.**

   a. **Select student responses to compare and contrast.**
   While scoring, select 2-3 different responses for the first three questions to use in the conversation. For each question, choose one response that correctly represents the probability as the ratio of target outcomes to total number of possible outcomes. Select other responses that suggest misconceptions or that reveal some difficulty thinking about fractional values, so that students have the opportunity to address these ideas. For the fourth question, select two different ways of partitioning the spinner that represents the ratios in questions 1-3.

   b. **Prepare questions to support and guide student thinking.**
   For example:

   Q: What strategy did you use to come up with your probabilities?
   Q: Come up with some comparative statements:
   It is more likely that____ than ____.
   It is less likely that ____ than ____.
   It is about as likely that ____ as ____.
   Q: What might happen if we did this again?
   Q: How did you label your spinner? What does it show?

   c. **Use an Assessment Conversation to help students consider probability as the ratio of target to all possible outcomes.**
   Invite students to present their responses. Guide the conversation with questions that direct the students to important elements of chance and chance models. For example, for Questions 1-3, when comparing a Cha (3c) response with a Cha (2b) response, it is important that students see what kinds of numbers to use in the numerator and denominator of the ratio and why. Similarly, for Question 4, when comparing a MoV(3a) response and an NL(ii) response, it is important to underscore part-whole relations and

what in particular is being compared (what are the parts? what is the whole?).

4. **Use Carnival 1 responses to generate a discussion of compound probability.**

   a. **Select student responses to compare and contrast.**
   While scoring, select 4 different student responses to question 1 and to question 2 for use in the conversation. If possible, these responses should include one from each level on the scoring guide. It is especially critical to compare the predictions obtained from combinations vs. permutations. Many students may prefer combinations. One way to help students is to visit whether or not the predictions made with combinations are the same as those made with permutations. To obtain predictions from the sample space, students form ratios of target outcomes to total possible outcomes. If necessary, prepare a TInkerPlots simulation to test predictions.

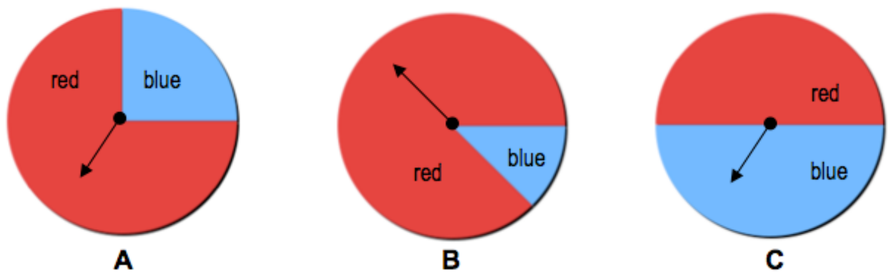
   b. **Prepare questions to support and guide student thinking.**
   For example:

   Q: How did you come up with these possible ways of getting a sum of X? Did anyone use a different way?
   Q: How did you calculate the probability of getting a sum of 5 or higher? Why did you X (make the decisions you made)?
   Q: What do you think might happen if we ran these spinners 100 times? About how many times would we expect to see a sum of 5 or more?

   c. **Use an Assessment Conversation to help students consider compound probability.**
   Invite students to present their responses. Guide the conversation with questions that direct the students to important elements of chance. For example, for Questions 1 and 2, when comparing a Cha (5c) response with a Cha (5b) response, it is important that students see why it is useful to consider permutations. It is also important that students understand how to use the sample space analysis to generate a probability. It may be helpful to expand the question to consider the probability of every possible sum.

## Exploring Expectation: Melissa's Spinners

Look at these spinners.




Melissa is thinking about what will happen if she spins each of these spinners again and again.

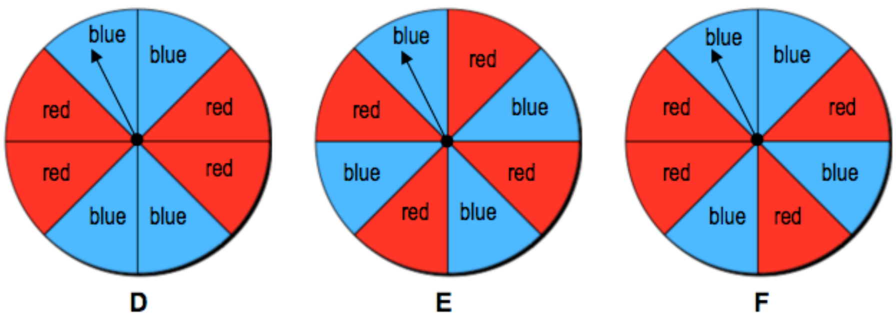
If, out of all the times she spins the spinner, she wants the spinner to land on red half the time, which of these spinners would be a good choice?

_______________________________________________

Why?

## Exploring Expectation: Sneaky Pete

Sneaky Pete decided that he did not want anyone to be able to know exactly how to draw his mystery spinner. So, he made his spinners like this and said that they would all give about the same results, even though they look different.



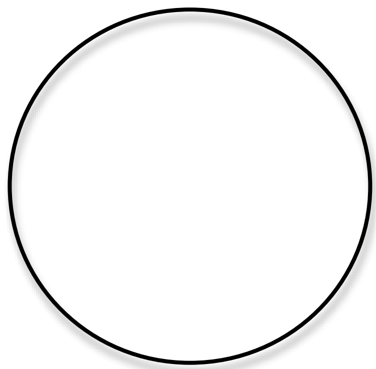

Do you agree with Sneaky Pete? ___________________________________

Why or why not?

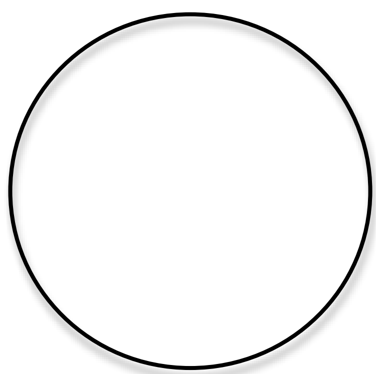
## My Mystery Spinner

Use TinkerPlots to design a 2-color, red-blue spinner. You must use 2 colors, and each color must cover at least 10 percent of the spinner. *Hide* the contents of the spinner.

Let your partner run the spinner 10 times. Ask your partner to draw his or her guess about its structure below.



Then let your partner run it more if she or he wants to. If your partner's guess changes, draw it below.



What did you notice?

## My Mystery Spinner (Blank Spinner)

Use the blank spinner and design a secret 2-color spinner. Make sure that you know how much of the area is covered by each color. For example, you might make ¼ of the area red and the rest black. But the smallest amount of area that can be covered by one color in any spinner is 1/10. After making the spinner, spin it 40 times and record the results on the table that is attached. Hide your spinner and share the results with a partner. Ask your partner to draw what your spinner looks like.

Was your partner right?

How close did he or she come?

What strategy did your partner use to figure out what your spinner might be?

Did you use the same strategy or a different one to figure out another person's mystery spinner?

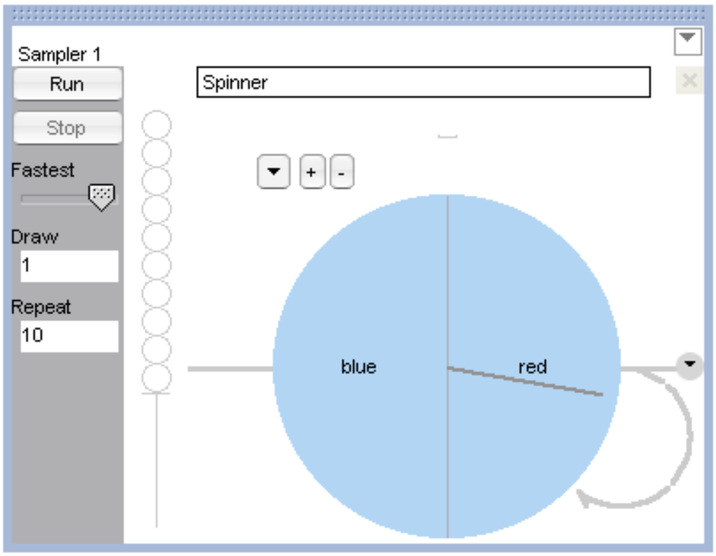
 

| Spin Number | Color Landed On | Spin Number | Color Landed On |
|---|---|---|---|
| 1 | | 21 | |
| 2 | | 22 | |
| 3 | | 23 | |
| 4 | | 24 | |
| 5 | | 25 | |
| 6 | | 26 | |
| 7 | | 27 | |
| 8 | | 28 | |
| 9 | | 29 | |
| 10 | | 30 | |
| 11 | | 31 | |
| 12 | | 32 | |
| 13 | | 33 | |
| 14 | | 34 | |
| 15 | | 35 | |
| 16 | | 36 | |
| 17 | | 37 | |
| 18 | | 38 | |
| 19 | | 39 | |
| 20 | | 40 | |

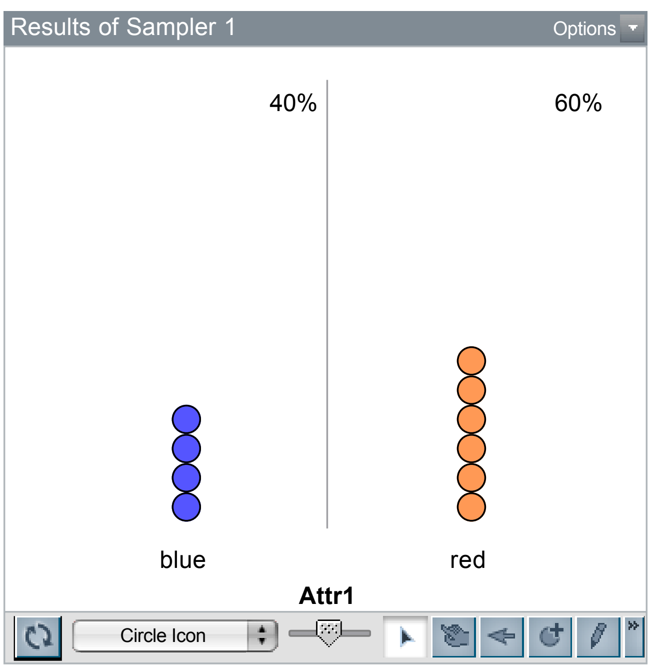
## Investigating Sampling

**10 Spins**

We will look at what stays the same and what changes when a chance process is repeated. Go to the Sampler menu of TinkerPlots and create a ½ "red" and ½ "blue" spinner. Set the number of repetitions to 10.




Construct a frequency plot of the outcomes and record the percent of each outcome.

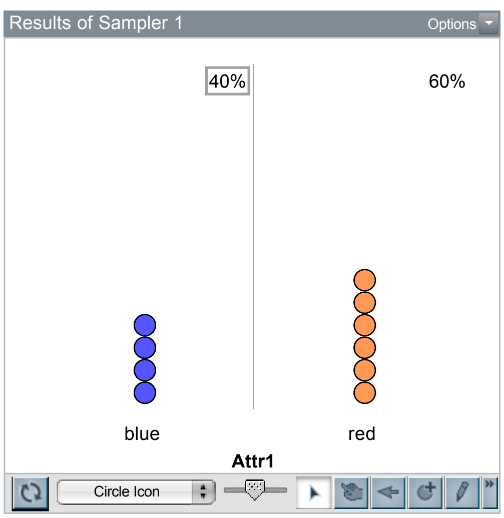


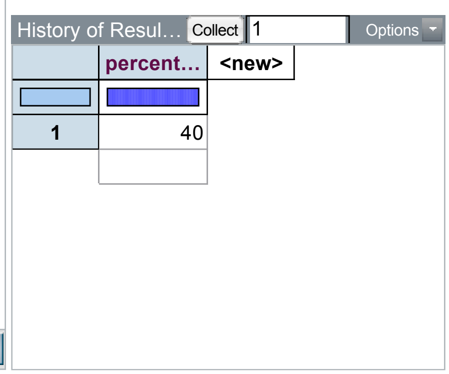

 

Run the sampler a few more times. What do you notice? What stays the same? What changes?

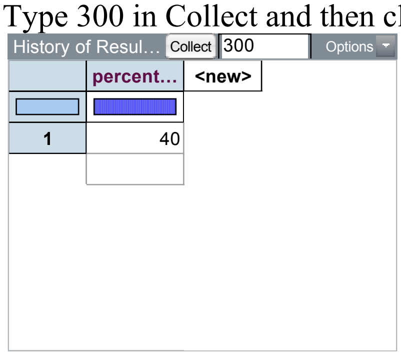
Why do you think this happens?

Now use TinkerPlots to collect 300 more samples of size 10. To do so, select one or both percentages, as shown here for percent blue, and then select "Collect Statistic" under the options menu. That will produce a History of Results window.

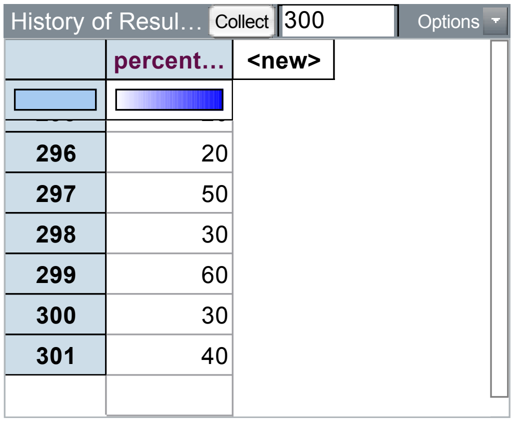


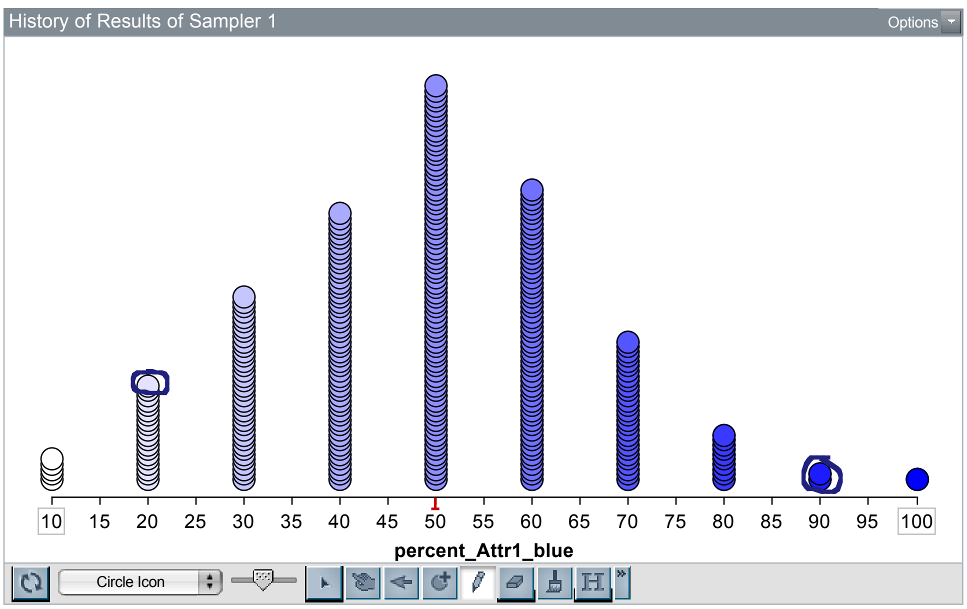

Type 300 in Collect and then click on Collect.



The result of the collection should look something like this:




Plot the Percent Blue. Your display may look something like this (without the drawing):




Answer the following:

How many spins of the spinner does each dot on the display represent?

How many outcomes were blue for highlighted sample on the left? On the right? How can you tell?

Why is the median percent blue for all of the samples 50?

**100 Spins**

Starting again with a 50-50 spinner, set the repetitions to 100. Each time, look at the percentage of blue outcomes. Construct a frequency plot of the outcomes and record the percent of each outcome. If you repeated this process of 100 spins 300 times:

What will stay the same?

What will change?

Why do you think so?

**Now run it 300 times.**

What did you notice?

What stayed the same?

What changed?

Why do you think this happened?

**1,000 Spins**

Starting again with a 50-50 spinner, set the repetitions to 1,000. Each time, look at the percentage of blue outcomes. Construct a frequency plot of the outcomes and record the percent of each outcome. If you repeated this process of 1,000 spins 30 times:

What will stay the same?

What will change?

Why do you think so?

**Now run it 300 times.**

What did you notice?

What stayed the same?

What changed?

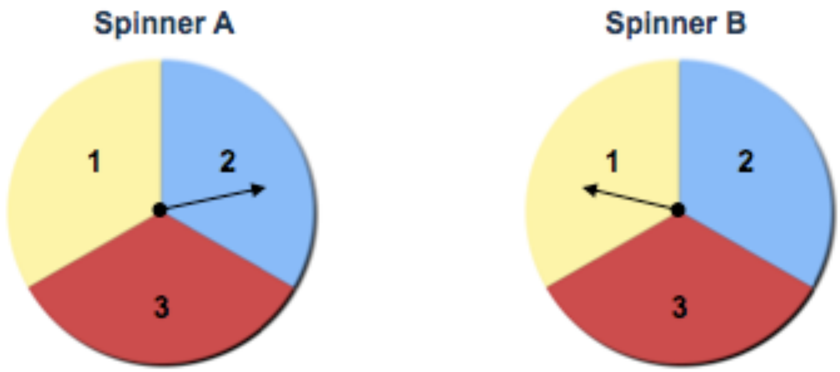
Why do you think this happened?

*Challenge:* Using one of the statistics that measures variability (or precision), come up with a way of comparing how the percentage of blue outcomes changes with the sample size (the number of spins: 10, 100, and 1,000).

## Investigating Compound Probability

Here are two spinners, each with three equal-area sectors:




**Possible Sums**

Which sums are possible?

Which ones are not possible?

**Find the Ways**

For each sum, list all the ways that sum could be formed. For example, a sum of 2 can be obtained by a 1 on Spinner A and a 1 on Spinner B.

**Make a Prediction**

Considering how each sum can be made, which sum will be most likely, or will all the sums be about equally likely? Why?

**Predicting the Results of 180 Times**

If you spun the spinners together 180 times, predict the number you expect to see for each sum. Record your prediction in the table. If you think of the number of 4's you expect and divide that by the total number of 2's, 3's, 4's, 5's, and 6's (180), we call that a probability. What is your guess about the probability of a sum of 4?

**Try it Out**

Run the spinners 180 times (if you are using TinkerPlots. If not, try it about 20 times and combine your results with those obtained from classmates). Record your results in the table. Considering probability of the sum of 4 again, what do your results suggest that it should be? What do you think accounts for the difference?

| Sums | Predicted (of 180) | Observed (of 180) |
|---|---|---|
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |

## Game on the Line

At the end of a game, your basketball team was ahead by one point. As time ran out, a player on the other team, Jena, was fouled attempting a 3-point shot. She gets 3 free-throw shots. Each free throw that she makes scores one point. For your team to win the game, Jena must miss all three shots. Over the year, Jena has made 50% of her free throws.

1. Without calculating it, what would you guess is the probability that Jena will miss all three shots and your team will win?

   a. 60% b. 50 % c. 25 % d. 12 % e. 5 %

2. In TinkerPlots, build a model that you can use to estimate the probability that Jena misses all three shots.

3. Make a graph of the data you collect. In your graph, order the different outcomes in a way that makes sense to you.

4. Based on your results, what is your estimate of the probability of Jean missing all three free throws?

   Estimated probability = ________

   Explain how you used the data from your model to estimate the probability.

5. How does the value you estimated from your model compare to what you predicted?

## Alex Rodriguez

Alex Rodriguez is the best player in baseball on the best team in baseball, the New York Yankees. Most years, Alex's batting average is .333 (he gets a hit about 1/3 of the time he bats).

What is the probability that Alex will get a hit three times in a row in the next game?

You can build a TinkerPlots mode to estimate this probability and/or you can just figure it out by considering all the possible outcomes. For example, the outcome we are most interested in is HHH (Hit, Hit, Hit) but another possibility is NNN where N stands for No-hit.

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# Unit Quiz



## Buttoned Shirts

A machine that puts buttons on dress shirts sometimes adds an extra button or leaves off a button. The company collected data about **20** shirts the machine made in an hour and **counted how often a button was missing or an extra button was added.**

Here is the data they collected:

| | **Missing a Button** | **Correct** | **Extra Button Added** |
|---|---|---|---|
| **Shirt Count** | 4 | 12 | 4 |

1. Given this data, what is the probability a shirt will be **missing** a button?

   ___________________

   Show how you calculated it.

2. Given this data, what is the probability a shirt will have an **extra** button?

   ___________________

   Show how you calculated it.

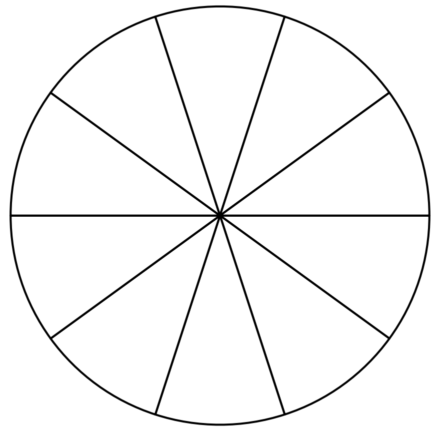
3. Given this data, what is the probability the shirt will have the **correct** number of buttons?

   ___________________

   Show how you calculated it.

4. Given the data, **label the spinner** on the right with the **probabilities** that the shirts will be missing a button, have an extra button, or have the correct number of buttons.

**Label the spinner by writing:**
**M** for missing buttons,
**E** for extra buttons, and
**C** for the correct number of buttons



# Unit Quiz



## Carnival 1


At the carnival, there is a slot machine. When you pull the handle, the numbers 1, 2, and 3 spin past both windows until the machine stops. Every time it stops, each window has an equal chance of showing the numbers 1, 2 or 3. Players win prizes depending on the numbers that show up in the two windows

1. One possible result of playing the game is shown above. The player got a 3 in the left window and 1 in the right. List all the other possible results

   **Answer**: (3,1),

2. A player wins a special prize if the sum of the two numbers is **5 or higher.**

   What are the possible ways to get a **sum of 5**?

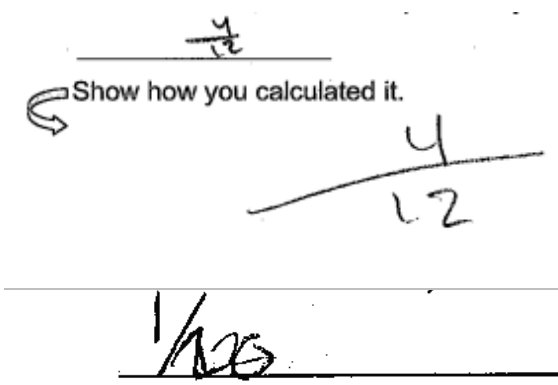
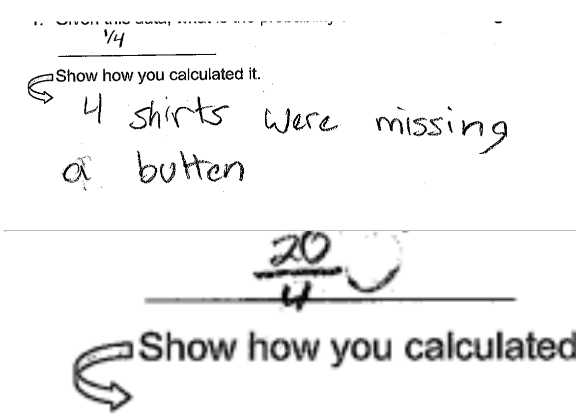
   What are the possible ways to get a **sum of 6**?

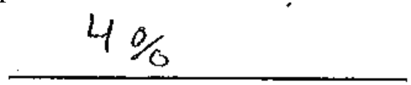
   What is the **probability of getting a sum of 5 or higher** and winning the special prize? Explain how you got your answer.

# Unit Quiz Scoring Guide — Investigating Chance Unit 5

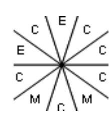
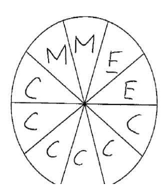
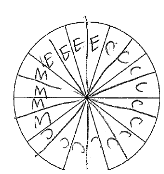
## Buttoned Shirts

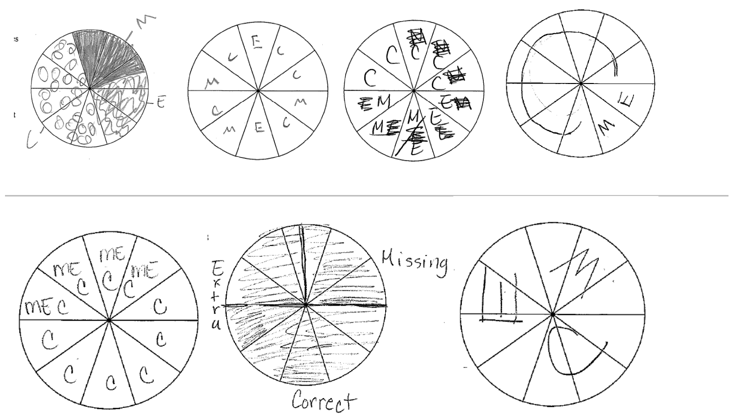
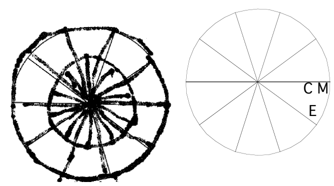
| Questions 1-3: Probability shirts will have a missing/extra/correct number of buttons<br>Buttoned Shirts and Chance | | |
|---|---|---|
| **Level** | **Performance** | **Example** |
| Cha(3c+) | 1. Correctly quantifies probability as the ratio of the number of target outcomes to all possible outcomes by providing the correct percentages or ratios for "Missing", "Extra" and "Correct."<br>2. Implies understanding of the notion of probability as a ratio based on an explanation or through clear calculations. (Students that provide very clear explanation and/or correct calculations but presents errors in mathematical notation can be included in this level if it is considered that those errors do not alter the interpretation of probability.) | 4/20<br>Show how you calculated it.<br>4/20 12+4+4= 20 8+12=20<br><br>1/5<br>Show how you calculated it.<br>4÷4 = 1 / 20÷4 = 5 4 missing / 20 shirts<br><br>4 out of 20<br>Show how you calculated it.<br>Missing a button 4 → 4/20 shirts 20<br><br>20%<br>Show how you calculated it.<br>4 + 12 + 4 = 20 4/20 = 1/5 = .2 = .20 = 20%<br><br>Example with error in mathematical notation:<br><br>a 1/5% chance<br>Show how you calculated it.<br>I added 4%+12%+4% then got 20%. 4% of the time a button is missing, so I rounded 4/20 to 1/5. |
| Cha(3c) | Correctly quantifies probability as the ratio of the number of target outcomes to all possible outcomes by providing the correct percentages or | 4 out of 20 |

| | | |
|---|---|---|
| | ratios for "Missing", "Extra" and "Correct" but no explanation or calculations are provided. |  |
| Cha(2b) | The response indicates that the student understands that the probability is a relationship between the frequency and a total, but the student is not able to correctly express the relationship by failing to correctly identify the numerator OR the denominator (the frequency). Student may use a consistent but incorrect value to represent the total. If a student makes mistakes both in the numerator AND the denominator he/she should be scored as NL(ii). |  |
| Cha(2b-) | The response indicates that the student understands that the probability is a relationship between the frequency and a total, but the student expresses the inverse relation, using the frequency as denominator. The student in this level may or may not have correctly identified the total (the numerator). |  |
| Cha(1b) | Provides the frequencies rather than ratios. All the frequencies are correct. A |  |

| | | |
|---|---|---|
| | student in this level may present errors in mathematical notation, such as the inclusion of a '%' sign, but the numbers accurately reflect the frequencies. |  <br>Example with error in mathematical notation:<br>  |
| NL(ii) | Relevant but incorrect response | • "M=5 shirts, E=5 shirts, C=10 shirts"<br>• "M=36, E=21, C=37" |
| NL(i) | Response is irrelevant, unclear, or a restatement of given information. | • "M=extra, E=correct, C=missing"<br>• "I don't know."* |
| M | Missing response | |

*Mock student responses

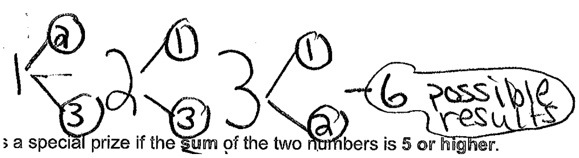
| Questions 4: Label the spinner, missing/extra/correct number of buttons<br>Buttoned Shirts and Modeling Variability (MoV) | | |
|---|---|---|
| **Level** | **Performance** | **Example** |
| MoV(3a) | **Use a chance device to represent a source of variability or the total variability of the system.**<br>May use letters, shading, patterns or arrows to label sections on the spinner. Labels sections on the spinner to represent proportions consistent with the data provided. |  *<br>  |

| | | |
|---|---|---|
| NL(ii) | Show some ideas in correct direction (i.e., vaguely relevant attempt). May recognize the relation between relative proportion of possible outcomes and their likelihood (shows evidence of part-whole relations) but makes an error on the graphical display. |   |
| NL(i) | Response is irrelevant, unclear, or a restatement of given information. |   |
| M | Missing response | |

*Mock student responses

## Carnival 1

| Question 1: List other possible results<br>Carnival Game 1 and Chance (Cha) | | |
|---|---|---|
| **Level** | **Performance** | **Example** |
| Cha(5c) | Student describes a sample space by listing ALL permutations of outcomes. | • Example 1:<br>Answer: (3,1) (3,2) (3,3) (2,1) (2,2) (2,3) (1,2) (1,3) (1,1)<br>• Example 2:<br>(3,1),(3,2),(1,2),(1,3),(2,3),(2,1), (3,3), (2,2), (1,1)<br>*Note:* (3, 1) is already listed, so students do not need to repeat it. However, students should list (1, 3) to get Cha5(c). |
| Cha(5c-) | Student describes a sample space by listing at least three possible permutations of outcomes, but missing others. | • Example 1: Student lists all but one possible permutation.<br>13, 31, 11, 33, 32, 12, 21, 22<br>• Example 2: Student lists most of the permutations.<br><br><br>• Example 3: Student lists some possible permutations.<br>3 \| 2<br>2 \| 3<br>1 \| 2<br>*Note:* (3,1) is already listed, so students do not need to repeat it. |
| Cha(5b) | Student describes a sample space by listing ALL possible combinations of outcomes, but does not consider permutations. | • Example:<br>(3,1) (3,2) (3,3)<br>(1,1) (1,2)<br>(2,2) |

| | | |
|---|---|---|
| | | Note: (3, 1) is already listed, so students do not need to repeat it. |
| Cha(5b-) | Student describes a sample space by listing at least three but not all possible combinations of outcomes. | • (2,3), (1,2), (3,1) *<br>Note: (3, 1) is already listed, so students do not need to repeat it. |
| Cha(5a) | Student constructs a new variable by joining outcomes of events (i.e., create compound/aggregate outcomes). The student lists one or two sums (or pairs). | • Example 1: Student lists one permutation of the example.<br>1L 3R<br>• Example 2: Student lists 2 pairs that are permutations of one another.<br>- (1,2) and (2,1) |
| NL(ii) | Response is relevant in that a) student gave values that came from the slot machine, but the responses do not show order or combination or b) the student includes additional responses that could not come from the slot machine. | • Example 1:<br>right 3 rong 1<br>• Example 2:<br>Answer: (3,1), (4,2) (3,2) (2,3) (1,2) (5,1)<br>• Example 3:<br>31, 13, 1,3, 0 |
| NL(i) | Response is irrelevant; no values are provided for the slots. | • Example 1:<br>there is 14 [illegible]<br>Answer: (3,1),<br>• Example 2:<br>They both spin Seast; they odd numbers. |
| M | Missing response | |

*Mock student responses

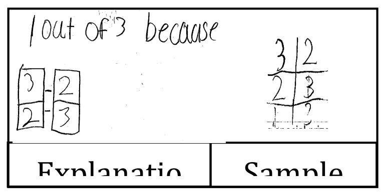
| Question 2a: Possible ways to get a sum of 5<br>Carnival Game 1 and Chance (Cha) | | |
|---|---|---|
| **Level** | **Performance** | **Example** |
| Cha(5c) | **Describe a sample space by listing *all* possible permutations of outcomes.**<br><br>The student lists (2, 3) AND (3, 2). | • Example:<br>2,3,32, |
| Cha(5b) | Student describes a sample space by listing all possible combinations of outcomes, but does not consider permutations. The student lists either (3, 2) OR (2,3). | • Example:<br>3+2=5 |
| NL(ii) | Response is relevant in that student gives spinner values, but are incorrect: Student either uses numbers from the spinners but summed to incorrect values, or used additional numbers not on the spinner. Student may also list probabilities instead of spinner values. Additionally, responses that talk about the "number of possible ways" are considered relevant. | • Example 1:<br>1&4 4&1 5&0 3&2 2&3<br>• Example 2:<br>(4,1)(1,4)(3,2)(2,3) 4<br>• Example 3:<br>NO possible ways |
| NL(i) | Response is irrelevant; no values are provided for each spinner. | • Example 1:<br>5/right 0/rong<br>• Example 2: |

| | | |
|---|---|---|
| | | 5, 55<br><br>• Example 3:<br>4/10 2/10 1/10 3/10 5/10 6/10 7/10 8/10 9/10 10/10 |
| M | Missing response | |

*Mock student responses

| **Question 2c: Probability of getting sum 5 or higher<br>Carnival Game 1 and Chance (Cha)** | | |
|---|---|---|
| **Level** | **Performance** | **Example** |
| Cha(6a) | **For compound (aggregate) events, relate sample space to probability.**<br><br>If student gets a probability of 1/3 (or equivalent), he/she is scored a 6(a) unless there is evidence showing that the student found that probability in a way that did not coordinate the complete sample space to the partial sample space. If the student said the probability was 1/3 and did not show any work, then he/she may be scored a 6(a). | three combows get 5 or higher and There are 9 total combows<br><br>3/9 |
| Cha(6a-) | **For compound (aggregate) events, relate sample space to probability.**<br><br>However, students either a) did not count | • Example 1: This student had 6 total outcomes and two that were equal to 5. Like other students, he assumed 5 or more meant sums equal to 5. Because of his explanation, we can assume that he is coordinating to the sample space. |

| | | |
|---|---|---|
| | all possible ways of getting 5 or higher (e.g. only counted the ways of getting 5) and/or b) the total number of sums is off. The students' response might equal 1/3 (the correct probability) but be obtained in a different way. Students who consider only combinations would actually calculate the probability as 2/3. | 2/6<br>I looked at the ways you can get 5 or higher and how many number combinations there are<br>• Example 2: Student had found 6 total outcomes for the sample space and he had noted 2 ways of getting 5 and 1 way of getting 6 so he only found probability for a sum of 5 rather than a sum of 5 or over.<br>2 of 6<br>• Example 3:<br>the special prize? Explain how you got your answer. The probability of getting a sum of 5 or higher is $\frac{3}{7}$ because there are 7 numbers and 3 are higher than 5 or higher. |
| Cha(3b) | Recognize the relation between relative proportions of possible outcomes and their likelihood. | • Example:<br>c. What is the **probability of getting a sum of 5 or higher** and winning the special prize? Explain how you got your answer. Not likly because there is not veng many wags to get 5 or a higher #. |
| NL(ii) | Students' responses are related to the question, but do not relate the outcomes to probability. For instance, student might provide incorrect probabilities that are not consistent with the outcome space they listed for question 1. Students who obtained the correct probability of 1/3 but not by coordinating the sample space to | • Example 1:<br>I think that the probibility of getting a sum of 5 or higher is 2 different ways.<br>• Example 2:<br>The probability of getting a sum of 5 is 3+2=5 or 2+2+1=5<br>• Example 3: |

| | | |
|---|---|---|
| | probability are also considered NL(ii). | (2,3)(3,2)(3,3) because 3+2=5 2+3=5 and 3+3=6 the higher numbers than 5.<br>• Example 4:<br>how you got your answer. or higher and winning<br>1 out of 3 Because there are three players<br>• Example 5: This students found the correct probability, but by a method not using his sample space. His explanation is shown on the left and the sample space he wrote in Part 1 is shown on the right.<br><br> |
| NL(i) | Response not related to the problem. For instance, student provides values that are not probabilities, e.g., greater than 1, or less than 0. | • Example 1:<br>I got my ansr by abbing 5 [illegible] times and I got 10 and so if you get one the four more and if you get tow moe you could wein.<br>• Example 2:<br>It is equil<br>• Example 3:<br>I didn't get the question? |
| M | Missing response | |

*Mock student responses